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The process of integration and the concept of integral: How does success with applications and comprehension of underlying notions such as accumulation relate to students' mathematical fluency

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THE PROCESS OF INTEGRATION AND THE CONCEPT OF INTEGRAL: HOW DOES SUCCESS WITH APPLICATIONS AND COMPREHENSION OF UNDERLYING NOTIONS SUCH AS ACCUMULATION RELATE TO STUDENTS' MATHEMATICAL FLUENCY

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At the end of the standard first calculus course, the student is expected to learn the Fundamental Theorem of Calculus, and to be able to use the integral to produce new functions, or numbers which, they are told, represent the "area under the curve". At the beginning of the standard second calculus course, students are expected to generalize their knowledge, and use the integration process to generate solids of revolution, surface areas, arc length and work, among other applications. Looking at students' success or failure in these endeavors, it was detected that there are marked differences in an aspect that, in this study, is called *mathematical fluency*.

The concept of mathematical fluency was developed, and the four parameters used in foreign language learning: reading comprehension, writing, speaking and listening comprehension were employed to measure mathematical fluency as defined in the present study. The types of mistakes made by the fluent and non-fluent students can be related to the types of mistakes made by the native or fluent speaker of a natural language, and those made by one who has not reached – or might never reach – that stage. The classification of *local fluencies*, with mathematical fluency as a global amalgam of these, was developed. Theoretical constructs such as Knisley's four stage model of mathematical learning, Tall and Gray's procept classification, Brousseau's cognitive obstacles as well as analysis of schemas, mental models and metaphors provided the language and concepts with which mathematical fluency, as detected by the four parameters, was described.

The study was realized through interviews, action research and observations of students in a second calculus course. In depth analysis using the four parameters of foreign language learning offers a methodology for studying student learning and understanding, that can be generalized to other mathematical areas, and adapted to quantitative as well as qualitative methods.

CHAPTER I

INTRODUCTION

"After all, we invent language, including formal languages like mathematics, to express our experience. It can be seen as a playful game. But anyone who has attempted to express something clearly and has struggled with language to make sure it is adequate to the silent thought to be expressed knows that language is the instrument of thought, sometimes even an obstacle to expression and not an easy game. Thought is simply not identical to language."

(Pagels, 1988, p. 23)

Rationale

Calculus is the gateway to advanced mathematical thinking. One of the reasons that calculus plays this role is that, to understand its content and possibilities, a shift in the way of thinking and doing mathematics is required. Although Newton and Leibniz are attributed with the discovery of calculus in the seventeenth century, the development of calculus started with the Greeks, and its content was not really formalized until the 1800's, due to what that shift in thinking and performing implies. Modern day calculus students must possess local fluency, as will be defined in the present study, in areas such as algebra, trigonometry and geometry, to be able to make the necessary connections that will permit success in applications, and understanding of underlying concepts; without these connections, the possibilities of using the powerful tools of calculus in a creative manner are reduced. The definition of mathematical fluency and its classification, as well as the method for detecting mathematical fluency using parameters employed in foreign language learning, is an important part of the present study. Without a method to detect where, in the process of learning, the problems lie, it will not be possible to correct them, or to design effective ways of transmitting knowledge. This study intends to shed light on aspects of the learning process related to applications of the integral, through the development of techniques for detecting mathematical fluency, based on the parameters of foreign language learning. The method itself transcends the particular case of integral calculus, and its validity for research in mathematics education in general is suggested.

The motivation for conducting research on the process and concept of integration, and relating it to mathematical fluency, is multifold. The concept of mathematical fluency, as developed in this study, is the backbone of the theoretical and methodological aspects of this work. The concept of mathematical fluency is closely related to fluency in natural language, and language learning. On the other hand, the subject of calculus as a whole, and integration in particular, is one of those places in the mathematics curriculum where many students, previously successful, begin to find insurmountable difficulties. The questions that will be raised in the present study are related to the learning process, and how this is affected by what students face in the standard second calculus program, as well as their personal background. Another factor which will be examined, in terms of learning, is how the initial presentation of the definite integral as a means for calculating the "area under the curve" could, in itself, be the cause of a "cognitive obstacle" (Brousseau, 1997), when the student is required to broaden that particular interpretation to the calculation of volumes, arc lengths and applications to physical concepts. The second calculus course should be a natural and coherent continuation of the first in which, according to the standard curriculum content, the Fundamental Theorem of Calculus and the Riemann sum approach to integration are learned. There are studies (Thompson, 1994; Czarnoch, Loch, Prabhu, Vidakovic, 2001) which show that this objective has not been achieved, and the present study intends to further explore the question of continuity and coherence in a particular phase of the calculus sequence. The present study will refer to the part of the second calculus course that deals with applications of the integral in terms of calculating volumes, arc lengths, surface areas, work and moments. The study is strongly oriented towards the understanding of how learning, when applied to integral calculus, is affected by the dual nature of the integral symbol itself, which can be seen as an instruction to carry out an operational process, as well as the embodiment of a specific object which is produced by that process, representing the mathematical concept of accumulation. In addition, this study examines how feasible it is to use the powerful techniques of integral calculus in a creative way (to do modeling), as a scientist, economist, statistician, or engineer, if conceptual understanding of the integral, as representing accumulation, is not achieved. Instructional methods and content, while not themselves the subject of the present work, will be analyzed and reflected upon.

Overview

Applications of the Integral

Calculus is an interesting subject to analyze from a learning and teaching perspective, given that it has so many facets. This study will focus on integral calculus, as it is usually taught in a standard second university calculus course, supposing the continuation of a standard sequence which means that differentiation, an introduction to integration through the ideas of antiderivative and Riemann sums, as well as the Fundamental Theorem of Calculus, were included in the first course, calculus I. Approximately the first half or more of the second calculus course typically concentrates on applications and techniques that use the integral as an operational tool to calculate areas, volumes of solids of revolution, surface areas of surfaces of revolution, arc length, and physical applications, as well as instructing the students in multiple procedures that rely strongly on their background in algebra and trigonometry. When dealing with the importance and nature of background, the relevance of mathematical fluency will be emphasized.

The very nature of the applications taught in the calculus II course, such as generating volumes and discerning between the disk and shell methods, call upon geometric intuition; in the same vein, a certain degree of analytic sophistication, plus strong algebra and trigonometric skills, are a requisite to be able to understand and operate with integration techniques. Added to this, as the student has already been exposed, in a first course, to the Fundamental

Theorem of Calculus and Riemann sums, it is expected that the processes mentioned above will be situated in a global context, in which the integral itself is understood outside of its varied interpretations, uses and manipulations, as an object that represents accumulation (whether of area, distances, volumes, work). The idea of the dual nature of symbolism in mathematics, as representing processes and concepts (Tall, 1994), is a natural reference frame. Language is an aspect of utmost importance as well, given that it is in the calculus context where the student sees "a new type of math from what we've learned all our lives" (Frid, 1994, p. 80). The symbolic language used in calculus, together with the new or deeper meaning given to familiar terms (Pimm, 1987), will be analyzed in this study, as related to the concept of integration. It is also of interest to test and see if the way that the integral and the integration process is presented in the first course, through Riemann sums and as representing the area under the curve, can convert itself into an obstacle in terms of the flexibility and capacity of generalization that the student needs to be successful in the study of integral calculus. It has been questioned (Czarnocha & Prabhu, 2001; Cordero, 1989) if the introduction of the definite integral as synonymous with the area under the curve could be counter-productive when attempting to expand this important, but particular, interpretation of the definite integral.

Mathematical Fluency

The concept of mathematical fluency that will be used in this study has its roots in other works (Bateman, Binder, Haughton, 2003; Layng & Johnson, 1992, NCTM Standards, 2000). However, it differs from the traditional definitions of fluency. In Chapter II, an overview of research and references to fluency will be given, and certain parameters related to the concept of fluency in foreign language that are considered relevant for the development of the concept of mathematical fluency in this study will be discussed.

The Standards (2000) of the National Council of Teachers of Mathematics (NCTM) define fluency as the use of methods that are "efficient, accurate and general." (p.32) In Chapter II it will be explained that the concept of fluency used by the NCTM differs from the one used in this study, although the definition of fluency in the present work is also stated in terms of efficiency, accuracy and flexibility. However, in the NCTM Standards, when talking about "computational fluency" it is stated that "Computational fluency should develop in tandem with the understanding of the role and meaning of arithmetic operations" and "understanding without fluency can inhibit the problem-solving process". (p.34) That is, fluency is seen as the possession of efficient and accurate methods, but not necessarily linked with comprehension itself, as it is in the concept of mathematical fluency used in this study. On the other hand, the concept of *local fluency* developed in this study, and explained in Chapter II, does subsume the idea, expressed in the Standards, of students becoming "fluent in arithmetic computation...fluent in computing with rational numbers in fraction and decimal form" and that "In grades 9-12, students should compute fluently with real numbers." (p.35). When the original concept of local fluency is explained in this study, it will be seen that it treats mathematical fluency in particular areas, whereas *global fluency* refers to the amalgams of local fluencies.

Finally, before entering into the actual study, it is important to situate this work from a personal perspective, and shed some light on how the research questions and general design were motivated. In mathematics education, when we see that a student gets a zero or a hundred, or that a student is efficient, accurate and flexible or not, we ask why? How do we measure fluency, understood as efficiency, accuracy and flexibility? I saw that question, and I began to think about fluency in foreign language learning. Now, of course, there is no isomorphism between mathematics and language. To begin with, no-one is a native mathematics speaker. But I thought about the four parameters of foreign language learning: speaking, writing, understanding (listening comprehension, reading comprehension), and I thought they might be useful in measuring mathematical fluency.

When students attend the calculus II class, or any class for that matter, they are usually exposed to spoken mathematics, which they are expected to understand (listening comprehension). Of course, they usually do not hear spoken mathematics outside of the class, unless they happen to go to office hours, or hear fellow students “talk math”, or listen to the videos that come with some books, which is not the usual case. This is similar to foreign language learners who are out of the language context, although it is more common for them to practice what they learned in class, with recordings. Reading and writing mathematics are practiced through homework, exams, and note-taking, while speaking mathematics is almost never exercised.

I asked myself what would happen if I were to take into account these four parameters, and relate what I call local fluency to the assimilation of new concepts, in particular the concepts which are needed to realize and understand applications of the integral. I decided to formulate a research design that would incorporate the four parameters of foreign language learning as a measure of mathematical fluency, while the concept of mathematical fluency itself also needed to be adequately defined. At the same time, I needed a technical language, based in mathematics education research, to express my findings. This technical language is presented in Chapter II as part of the theoretical framework of this study; the other part of the theoretical framework consists in the development of the concept of mathematical fluency, and how it can be measured using the four parameters. The design was developed for qualitative research, but can be extended to large scale quantitative research as well, as will be mentioned in Chapter VI.

Research Questions

The research questions address the learning of applications of the integral as introduced in the calculus II course, and how this learning manifests itself, and is different, in fluent and non-fluent students. The actual content, meaning, justification and classification of concepts such as fluency, metaphors and cognitive obstacles, among others, will be treated in Chapters II, III and IV, in which the theoretical framework, literature on calculus learning, and methodology, which includes measurement criteria, are explained and developed. To better understand the research questions when presented in this introduction, it is important to mention that the parameters that measure mathematical fluency, as defined in this study, are the same parameters used in foreign language learning that is, listening comprehension, reading comprehension, speaking and writing. Performance is measured in terms of the student's success or failure with the specific interview questions that are presented in Chapter IV, as well as the material presented in the classroom context.

Research Questions With Respect to Mathematical Fluency

1. What is the relation between students' comprehension of written mathematics (reading comprehension), and mathematical fluency?

2. What is the relation between the logical structure, method of attacking problems, sequence of steps and sketches in students' written mathematics, and mathematical fluency?

3. What is the relation between students' spoken mathematics and mathematical fluency?

4. What is the relation between students' comprehension of spoken mathematics (listening comprehension) and mathematical fluency?

Research Questions with Respect to Performance:

Applications

1. What stage(s) does the students' reasoning reflect when dealing with applications?

2. What are the key metaphors used when having to deal with applications?

3. What role do cognitive obstacles play when students are asked to set up and use the integral for applications different from the area under the curve?

Fundamental Theorem of Calculus.

1. How do students deal with the process-concept duality when confronted with the symbols of this part of calculus?

2. What stage(s) does the students' reasoning reflect?

Chapter II establishes the theoretical framework implicit in the present study. Chapter III gives a panorama of relevant literature on calculus learning. Chapter IV explains the context and methodology employed in the study. Chapter V presents the results of the study, and Chapter VI presents conclusions and implications of the results.

CHAPTER II

THEORETICAL FRAMEWORK

When conducting a study in any discipline, it is important to build upon previous work done in the field. In the case of pure and exact sciences, this fact is built into the research itself, given it is impossible to do a research project in, say, pure mathematics in which procedures and theory are not the result of years of previous study, and the actual problem is not part of a sequence. In the social sciences, while this is also true, it is necessary to be explicit that the work itself is part of the evolution of the area. In the context of mathematical education, which is a sort of hybrid between the precision and techniques of research in pure mathematics, and the more qualitative techniques of research in cognitive psychology and education, it is necessary to expound upon the theoretical framework which will be used in the research process. By no means does this imply a dogmatic treatment of the subject; on the contrary, if the theoretical framework is well defined, it is easier to detect any conflicting or reinforcing evidence.

The theoretical framework will include the concept of *procepts* developed by Gray and Tall (1992), as well as the idea of the dualism *process-object* as developed by Sfard (1991). A f*our-stage model of mathematical learning,* as constructed by Knisley (2000), is an important part of the theoretical framework of the present work. The concept of *Abstraction,* from different perspectives

(Dubinsky, 1991; Lehtinen and Olhsson, 1997; Harel and Tall, 1991) is discussed, as well as the difference between *concept definition* and *concept image* as expressed by Vinner (1991). A brief exposition of the cognitive theory of *mental models* is presented, as well as a section on certain relevant *references to linguistics*. The concept of *cognitive obstacles,* as developed by Brousseau (1997), is also included in the theoretical framework. Finally, there is a section on *mathematical fluency*, in which different definitions of fluency in the literature are discussed, before entering into the definition of the concept of mathematical fluency as employed in this study.

Procepts

David Tall, together with Eddie Gray, over the last 15 years have been developing a theoretical framework in which it is possible to answer the question "Why is it that so many fail in a subject that a small minority regard as being trivially simple? (1992i). They have come to the conclusion that "the more able are doing qualitatively different mathematics from the less able." (1992i). Their focus is designed to give greater insight into the use of symbols in mathematics, and how they play a dual role in terms of *process* and *concept*. Gray & Tall's definition of process is "a *mathematical* process represented by the mathematical symbolism". (Davis, Tall & Thomas, 1997). For example, in the context of Gray and Tall's *procept* theory, they consider the

> *duality between process and concept* in mathematics, in particular *using the same symbolism* to represent both a process (such as the addition of two numbers 3+2) and the product of that process (the sum 3+2). The *ambiguity of notation* allows the successful thinker the *flexibility in thought* to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema.

In their landmark article in 1991, they stated that they "...hypothesize that the successful mathematical thinker uses a mental structure which is an amalgam of process and concept which we call a *procept*". (1991). Since that article, they have refined and tested their hypothesis, including a gamut of mathematical thinking that goes from early childhood counting, to advanced mathematics such as Calculus, Linear Algebra, Analysis, Differential Equations, Proofs, etc. In this vein, they comment that

> In the earlier stages of mathematics, the associated processes are given by explicit procedures, such as counting, use of multiplication tables, algorithms for multidigit arithmetic, and so on. But procepts in the sixth form will be found to include processes such as “tending to a limit” or symbolic integration where there may not be a simple procedure to carry out the process.

As an example they mention that “the integral $\int f(x)dx$ evokes both the process of calculating the integral and the symbolic function produced by this process.” (1997i).

The Dualism "Process-Object"

In 1991, Ana Sfard wrote "On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin," (Sfard, 1991) where the author asks herself a rhetorical question: "How does mathematical abstraction differ from other kinds of abstraction in its nature, in the way it develops, in its functions and applications?" In Sfard's framework, the historical development of mathematical concepts and the individual's learning of mathematics are similar; several mathematical concepts are taken (negative numbers, rational number, complex numbers, functions) to illustrate how, in the beginning it is the operational approach that appears. For example, the repeated observation of children in the process of learning how to count, shows that there is a stage in which, as an answer to the question of "how many numbers are there", the child will answer by counting "one-two-three". Other examples, such as the learning of complex numbers, show that first the symbols are no more than placeholders or abbreviations for certain operations, and it takes a while for real understanding to develop, and for the complex numbers to be conceived of as abstract objects, with their own "identity". Sfard systematizes the process of passing from an operational approach to a structural approach by means of stages. These stages, once a mathematical concept is presented, are classified as *interiorization, condensation,* and finally *reification*, this being the stage in which the structural understanding is achieved. In this framework it is emphasized that the operational and structural approaches are complementary. It is important to follow algorithms, such as the evaluation of functions, and it is important to compute; it is suggested that a structural understanding is based upon a previous operational familiarity with the concept. At the same time, a purely operational approach is not practical, given the fact that "the distance between advanced computational processes and the concrete material entities which are the objects of the most elementary processes (such as counting) is much too large to be grasped by us in its totality." (p. 29) Another strong argument that the author makes is that the transition from processes to abstract objects enhances the sense of understanding mathematics. Psychologically, it can be argued that without reification, there is a sense of incompetence, or lack of understanding, that provokes insecurity, even when only the operational (process) aspect is being evaluated. Even professional mathematicians, when honest, have admitted that sometimes the real meaning of the ideas they are manipulating does not come in an easy way. How to stimulate reification is a question that needs to be researched, from a general cognitive point of view as well as from a contextual one.

One very interesting aspect about the particular subject of integrals as linear operators (although they are not presented as such to the student) is that the integral sign, an "extension" of the sigma notation for sum, demands that the operation be performed on functions; this means that, if the student has not been able to reach the reification stage for functions themselves, and perceive them as objects, it might be very difficult for the student to really understand the level at which he is working when he performs the operation on functions and arrives at another function as a result. This specific problem has been documented in the literature (Czarnocha, Loch, Prabhu and Vidakovic, 2004; Dubinsky & Harel, 2001; Sfard 1991). According to Doorman and Gravemeijer (1999)

> the graphs of functions develop into *models for* formal mathematical reasoning about calculus. In relation to this we can speak of a process of reification. However, it is not the graph that is reified, it is the activity that is reified, that is, the act of summing is reified, and becomes the mathematical object "integral". (pp. 122,123).

A Four-Stage Model of Mathematical Learning

In a very interesting paper, J. Knisley (2000) develops a model of mathematical learning to provide a rationale for the new calculus text he was writing at the time; this author categorizes four stages of mathematical learning. The first stage, *allegorization,* draws on the idea of analogy: "a new concept ... described figuratively in a familiar context in terms of known concepts". The other stages are *integration,* in which "comparison, measurement, and exploration are used to distinguish the new concept from known concepts", *analysis,* where "the new concept becomes part of the existing knowledge base", and *synthesis,* in which "the new concept acquires its own unique identity and thus becomes a tool for strategy development and further allegorization."

In this framework, the "learning style of a student is a measure of how far she has progressed (through these four stages)". The *allegorizers* "cannot distinguish the new concept from known concepts", the *integrators* "realize that the concept is new, but do not see how the new concept relates to familiar, well-known concepts", the *analyzers* see the relationship of the new concept to known concepts, but lack the information that reveals the concept's unique character, and the *synthesizers* have mastered the new concept and can use it to solve problems, develop strategies (i.e., new theory), and create allegories." Although allegorization (and the allegorizer) seem to be on the bottom rung of the ladder, this scheme is not at all rigid, and we can think of it more as an interdependent whole, in which “learning a new concept begins with allegory development”; once it is integrated into the existing knowledge base, analyzed and synthesized, it becomes a tool that can be used to develop strategies which will, in turn, be used to develop allegories of new concepts. In the context of the present study, this categorization can be helpful as a complement to the procept and process-object frameworks; I think that the author comes upon and shares a very important insight when he explains the analysis stage.

> In short, analysis means that the student is thinking critically about the new concept. That is, the new concept takes on its own character, and the student’s desire is to learn as much as possible about that character...As a result, analyzers desire a great deal of information in a short period of time, and thus, it is entirely appropriate to lecture to a group of analyzers. Unfortunately, the current situation is one in which we assume that all of our students are analyzers for every concept, which means that we deliver massive amounts of information to students who have not even realized that they are encountering a new idea. (p. 9)

This four-stage model of the learning process of new concepts is employed in a very precise way in the present study. The research questions include the classification of the students that participate in the interviews in terms of their stage in the learning process of the concepts that are introduced. These concepts include applications of the integral (finding volumes, arc lengths, surface areas, work) as well as the concept of accumulation, which was introduced with The Fundamental Theorem of Calculus in the first calculus course. It should be mentioned that the classification of a student in a particular stage is, in this study, concept-specific.

Abstraction

The stages of *reflective abstraction*, as developed by Dubinsky (1991) are also applicable to the concepts of integration and integral. According to Dubinsky "reflective abstraction appears as a description of the mechanism of the development of intellectual thought." (Dubinsky, 1991). There are specifically five categories that come from the Piagetian classification of the development of children's logical thinking, and Dubinsky applies these categories to the context of advanced mathematical thinking "to describe how new objects, processes and schemas can be constructed out of existing ones." These categories are *interiorization, coordination, encapsulation, generalization* and *reversal.* The first three are very similar to Sfard's classification, with encapsulation playing the role of reification; the other two are related to the interplay between abstraction and generalization, which will be explained below. In terms of integral calculus, Dubinsky uses this framework when he notes that

> The indefinite integral forms an important example that can be interpreted as encapsulation together with interiorization. Estimating the area under a curve with sums and passing to a limit is, of course, a process. Students who seem to understand this often have difficulty with the next step of varying, say, the upper limit of the integral to obtain a function. What is lacking, we suggest, is the encapsulation of the entire area process into an object which could then vary as one of its parameters vary. This would then form a "higher-level" process which specifies the function given by the indefinite integral.(p. 105)

At the same time, the first three stages, interiorization, coordination and encapsulation form part of the four part theory, known under the acronym APOS

(1991) in which the *action,* after being repeated, will turn into a process (ideally), then will be encapsulated as a mental object and will form part of a *schema.* The schema is a concept that comes from cognitive psychology, often employed in computer science, and can be defined as general knowledge structures that form long term memory and encapsulate numerous elements of information into a single element, organized in a manner in which it can be widely used.

Tall and colleagues in their articles from 1997 and 2000, both titled "What is the object of the encapsulation of a process?" address the importance of the "encapsulation" (Dubinsky 1991) or "reification" (Sfard 1991) of a process , and begin their articles under the subheading **Theories of encapsulation/reification** where they review the literature on this subject. They trace the encapsulation/reification concept's origin to Piaget, who noted how "actions and operations become thematized objects of thought or assimilation." They also show how Dienes (1960) and Davis (1984) "used a grammatical metaphor to formulate how a predicate (or action) becomes the subject of a further predicate, which may in turn become the subject of another" (This aspect will appear again in this section under the subheading of *References to Linguistics*). They also mark the differences between their procept construction, and the process-object/encapsulation construction, especially in terms of the scope:

> The scope of *procept* theory is narrower. This does not mean that it is weaker, since it is designed to give greater insight into the profoundly powerful use of symbols in mathematics to switch effortlessly between concepts to think about and mathematical processes to solve problems...The notion of *procept* was never intended to have the same broad scope as the theories of Sfard or Dubinsky. Neither the child's notion of permanent object nor the students' notion of axiomatic system are procepts because neither has a symbol capable of evoking

> either process or concept. Nor is the notion of procept defined to be explicitly tied to the situation in which the mental object represented by the symbol is necessarily construction by 'encapsulation' from the corresponding *mathematical* process, even though this is the way in which many procepts are constructed." (Davis, Gray, Tall, Thomas, Simpson, 2000, p.11)

Then, referring to their own work (Gray & Tall, 1991, 1992,1997) they emphasize that "Gray & Tall's notion of *procept* only occupies part of this scenario because their notion of "process is a *mathematical* process represented by the mathematical symbolism."

Another approach to abstraction in general, and mathematical abstraction in particular, is taken by Lehtinen and Ohlsson (1997). These authors contrast the standard claim that abstraction implies generality, with the cognitive science position that knowledge is domain-specific and particularized. Their intent is to frame abstraction from a different perspective, neither the standard nor the predominant cognitive science perspective (although they, themselves, are cognitive scientists), one which "turns the relation between similarity and generality on its head". (p. 41). When the authors talk about inverting the concepts of similarity and generality, they are referring to the following: whereas in the classical perspective similarities are the basis for generalization, here the recognition of similarity is based on abstraction that is possessed beforehand. The authors insist on the falsity of the way diverse concepts, theories, and even laws discovered in science and mathematics have been historically interpreted. The mere act of observing hundreds of equilateral triangles, according to this perspective, would not provoke recognition unless the person already has the idea of an equilateral triangle. In the classical tradition "Generality is the *product* of learning. In the present view...abstraction is a *prerequisite* for learning." (p,43). In this context, the learning of a deep idea depends on assembling existing abstract ideas into a new and more complex configuration. "The assembly process moves from the simple toward the complex, not from the concrete toward the abstract". (p.42) The authors claim that "The regrouping of the appearances on the basis of what abstractions they fit, as opposed to by what perceptual similarities they exhibit, is the main lever by which human beings pry open the secrets of nature." (p.47) An example of this statement could be the obvious fact that the bacteria in a test tube are not perceptually similar to a consumer market; nevertheless, it is very possible that the same growth function could model both situations!

Tall and Harel (1991) distinguish between *generalization* in the sense of just expanding and extending familiar processes, and *abstraction*, another type of generalization for these authors, in which mental reorganization is required. These authors call

> ...an *expansive generalization* one which extends the student's existing cognitive structure without requiring changes in the current ideas.... A generalization which requires reconstruction of the existing cognitive structure we call a *reconstructive generalization*. (p. 12).

For example, the notion of integral as an accumulation, and its relation to the derivative through the Fundamental Theorem of Calculus does require reconstructive generalization; that the integral can be used to calculate volumes and arc lengths as well as area, due to the fact that all are infinite sums, is an expansive generalization.

When conducting any type of analysis of the learning or teaching of mathematics, it is necessary, and unavoidable, to discuss abstraction, as mathematics itself is generally referred to as an abstract subject. Although abstraction, as such, will not be measured in the present study, it is an implicit factor in the performance of calculus students.

Concept Definition and Concept Image

A formal mathematical definition is something with which any mathematics professor must work, whether or not it is expected that the student will also force his reasoning to revolve around the definition, if it is given to him in a formal way. The usual second calculus course in the United States does not place a major emphasis on the students' work with definitions (or theorems, for that matter); however, even in courses where the definition is part of the students' responsibility, the definition itself is often learned by memory, and repeated as such, by rote, on the evaluations. On the other hand, to determine if a definition has been really understood, it must be seen if it is applied when the student is put to work on particular problems.

Vinner (1991) differentiates between the definition of a concept ("concept definition") and what he calls the *concept image.* "The concept image is something non-verbal associated in our mind with the concept name". (68). According to this author, "To know by heart a concept definition does not guarantee understanding of the concept. To understand, so we believe, means to have a concept image. Certain meaning should be associated with the words". This does not necessarily mean that the concept image is complete, or even right, which in itself is another kind of problem.

Mental Models

A mental model is not constructed in a vacuum of previous experiences and knowledge. It is evident that a theory of mental models and representations has to take into account the way that the mind will incorporate previous knowledge, and must distinguish between those models that link the previous knowledge with new challenges in a logical, or even successful way, and those that are not capable of doing so. According to Johnson-Laird

> Mental models emerged as theoretical entities from my attempts to make sense of inferences, both explicit and implicit. They replaced the formal rules of a hypothetical mental logic. Subsequently, I was able to give a better explanation of meaning, comprehension, and discourse, by postulating mental models in place of other forms of semantic representation. (1983, p. 397).

This way of explanation doesn't define mental models, although it gives us an idea of their importance in the context of certain kinds of research. A common way of referring to mental models is as "representations that are active while solving a particular problem and that provide the workspace for inference and mental operations". (Halford, 1993, p.23). Although there is by no means a consensus on the use of the term, it is common to implicitly or explicitly acknowledge certain constraints on any possible models. When investigating mental models in any particular context (in the case of the present study, integral calculus), the cognitive representations of the problem at hand are observed by means of speaking, written symbolic work or sketches. This permits, in the case of mathematics education (as mental models are a construction of general cognitive psychology), the analysis of how the student organizes his schema in the process of dealing with specific problems in a specific context.

English and Halford (1995) state that

> Mental models can guide the development of strategies and acquisition of cognitive skills ...Once the strategy is developed ... (the) mapping is no longer required, and processing loads are considerably reduced. Thus the development of skills based on understanding can be expected to require allocation of high levels of resources. Application of the strategies once they are developed is much less effortful. (1995, p. 49).

This observation is very important for mathematics education in general. In terms of human reasoning processes, these authors affirm that "Human reasoning processes depend more on memory retrieval and analogy than on application of formal logical rules". I will come back to the term *analogy* in the subsection on linguistics.

Another concept that is useful when dealing with mental models is that of a *cognitive system.* A cognitive system "comprises a problem, a symbolic representation of that problem, and the relations between the two". (Halford, 1982, p. 74). The symbol system is expressed as a set of mappings, and defined as a set of elements and a function defined on those elements. As there are *symbol systems*, the physical environment of a person also consists of systems, *environmental systems,* which are also defined by mappings. A cognitive system must consist of a symbol system that is isomorphic to an environmental system, and is expressed through a *commutative diagram*, a mathematical concept from category theory, as follows:

$$
\begin{array}{ccc}
S,\ldots,S & \rightarrow & S \\
\downarrow & & \downarrow \\
E,\ldots,E & \rightarrow & E
\end{array}
$$

What this means is that symbolic elements can be mapped one-to-one on environmental elements, and then mapped through the function defined on the environmental system or the symbolic function can be employed and then mapped to the environmental element on the bottom right corner. It is important to point out that the symbol system, according to the theory of mental models, is stored in the long term memory, and called upon, for example, in the problem solving process. Another aspect that should be mentioned is that the symbolic processes must be structurally isomorphic to the problem environments to which they are applied, and must be applied in such a way that all mappings of symbols to environment are consistent or invariant within any one application. However, the symbolic processes by no means need to be a copy of any environment with which they interact. Mathematical modeling is an example of a process which can be successfully analyzed using a cognitive system. A simple example could be linear depreciation, in which the symbolic system comes from the study of the line, and the environmental system originates from a particular case with a particular item.

Tall (2005), when describing aspects of the brain that "enable us to build a highly sophisticated mathematical mind", refers to the *compression* of ideas into "thinkable concepts that can be held in the focus of attention". (p. 1). The compression of mathematical knowledge has been described by several of the working mathematicians who have reflected upon their own processes. The compression of mathematical ideas is described in the following quote by Poincare: "...I can perceive the whole of the [lengthy mathematical] argument at a glance. [Thus] I need no longer be afraid of forgetting one of the elements; each of them will place itself naturally in the position prepared for it, without having to make any effort of memory." (Sfard, 1991, p. 29). According to Tall, "the general biological faculty that enables this to happen is the strengthening of links between neurons that prove successful and the suppression of others that are less relevant, building mental modules that operate together in consort in a process termed *long-term potentiation.*" (p.2). The compression of mathematical ideas into thinkable –thus "workable"- concepts is what also permits a successful development of flexibility, as well as efficiency (Tall, p.6) when dealing with symbolic operations, and is essential to mathematical fluency, as I will explain later.

Mental models, schemas, strategies and compression of mathematical ideas are a fundamental component of the language and concepts of analysis in the present study. One of the principal objectives of the present work is to use the four parameters of foreign language learning (listening comprehension, reading comprehension, speaking and writing) to measure fluency, defined as efficiency, accuracy and flexibility. The strategies used by the interviewees in attacking and solving the specific questions related to applications of the integral reflect the degree of efficiency and flexibility of the student, two components of fluency, as measured by the parameters. These strategies are product of the schemas and mental models possessed by the student, as well as the ability to compress ideas.

References to Linguistics

Mathematics has often been referred to as a language (in the same way that written music has also been); at the same time mathematics, despite possessing its highly developed symbolic language, is taught and written in vernacular language, especially when the conceptual aspect is emphasized, such as when writing proofs that are not algorithmic. When language is used, say in teaching and explaining new mathematical concepts, "figures" of speech and vernacular language are mixed with precise definitions and "pristine" expressions. Sfard (1997) distinguishes between *metaphor* and *analogy;* "Analogy enters the scene when we become aware of a similarity between two concepts that have already been created; the act of creation itself is a matter of metaphor." (p. 345). The author also mentions *structural* metaphors "which renders the target concept the characteristic of an *object,* as opposed to the operational metaphor which makes it into a *process.*" (p. 351). All of this framework is used to reflect on the pedagogical implications in mathematics of the process of metaphorical conceptualization. It is given that "...metaphor is viewed here as the essence of conceptualization, learning would clearly be inconceivable without it." However, in terms of the pedagogical value of metaphors, there is no definite answer; some metaphors can be valuable in terms of promoting understanding, while others can clearly be prejudicial and serve as obstacles to future learning or generalization. The metaphor "while being the very cognitive device which makes our conceptual thinking possible, also constrains our imagination". (p. 369). There are examples from the history of mathematics which are very illustrative, such as $\sqrt{-1}$, which was called an *imaginary* number. This naming, which is an extra-mathematical metaphor (described in the next paragraph), was not helpful at all in terms of conceptual understanding or development of the complex number system. It was a misleading label, nothing more, until it was situated as a point in the complex number plane. Sfard talks about the role of imagery, saying that

> Metaphorical projection...involves both a transfer of linguistic templates and the transmission of...image schemas...They are two different facets of the same story -the story of construction of a new meaning...it is hardly surprising that imagery may be a key to high-level mathematical performance. (1997, pp. 359,360).

Primm (1987) distinguishes between *extra-mathematical* metaphor and *structural metaphor.* His definition of structural metaphor will be the one employed in the present study. The extra-mathematical metaphor refers to objects and concepts outside of the mathematical realm, which are used in the teaching process to "help" create a mental image. This is the case, for example, of "right" triangle, as the Greeks considered this triangle the epitome of correctness in the triangle world, and has nothing to do with the angles being on the right or left-hand side! On the other hand, structural metaphors come from mathematical language itself, and are exemplified by expressions such as the *slope of a curve* at a point, or the *complex plane* when a complex number is really a scalar. Metaphors can be very important when realizing qualitative research, because often it is necessary to detect and then interpret either the metaphor itself, or the understanding – or confusion – that it may provoke, as will be explained in the methodology section, and exemplified in the section on results. It suffices to say that, in the case of the aspects of integral calculus studied in this work, especially in the part related to the generation of solids of revolution, surface areas, arc lengths and applications in general, it was very important to detect the *image schemas,* and their "translations" to spoken language of the students who participated, as much of the decision making in terms of setting up the integral was based on their sketches and graphs.

Ferrini-Mundy and Graham (1994), in their research on calculus learning, find that " 'Natural' language is used readily in describing mathematical ideas and appears to be more useful than formal definitions or vocabulary....Sandy's (one of their interviewees) concept of continuity...is described repeatedly with language such as 'it just keeps going', 'has no definite end', and 'it all flows together' ". (p. 43). Their "natural language" coincides with Primm's extra-mathematical metaphors.

Presmeg (1997) distinguishes between *metaphor* and *metonymy.* Metonymy is a much less familiar concept, and is defined loosely as "a figure by which one word is put for another on account of some actual relation between the things signified ... Metonymy is the basis of all mathematical symbolism. For instance, 'Let x be an integer...', 'Let ABC be any triangle...', any mathematical statement in which a symbol stands for a class, a principle or some mathematical concept, is a statement which uses metonymy." (pp. 270,271). Presmeg declares that the reasoning process with metaphors and metonyms, in which "metaphoric signification (for meaning) and metonymic signification (for symbolism) are both essential components in the apprehension of mathematical structure." (p. 271) She points out that teachers and students alike use metaphor and metonymy in teaching and learning mathematics.

Semiotics, which is the study of signs and signifying practices, tries to understand how certain structures produce meaning, rather than work on the subject of meaning itself. Saussure's Swiss structural approach (Whitson, 1994), substitutes the *sign* with a combination of a *signified* and a *signifier*, "This substitution... also generalizes his definition of the sign beyond his initial reference to *linguistic* signs (with sound patterns as signifiers), so that he could now propose a more extensive new social science of "semiology"". (pp. 38, 39). Saussure's theory was modified by Lacan who developed the concept of the *chaining* process of signifiers, ".. in which the signifying term..." (for example, a person's name) "... in a preceding sign combination comes to serve as a signified term in a succeeding sign combination" (for example, when the name is represented by a number, because what's important is the number of people, not the particular person with his or her name). It is also important to understand that, in mathematics, it is very common that a signifier may stand for very different signifieds. For example, in abstract algebra the operational signs of + and • can signify very different binary operations than addition and multiplication as they are commonly thought of. Returning to the concepts of metaphor and metonymy, it is "...imagery which often helps to supplement the signifiers with signified and to turn their use, eventually, from templates-driven into *semantically mediated*." (Sfard, 1997, p.360). Also, Presmeg states that

> If all symbolism in mathematics involves signifiers and signifieds in various relationships, then all of mathematics must involve metonymy, as there is no branch of mathematics where a system of signification is not used. Successive chaining of signs involves reification, and progressive levels of abstraction. (p. 277).

The detection and analysis of the appearance and effect of extra-mathematical and structural metaphors was considered part of the methodology of the present study, and is explained in Chapter IV. In Chapters V and VI, the results are shown and implications are drawn.

Cognitive Obstacles

The concept of cognitive obstacles, as developed by Guy Brousseau (1997), consists of several parts. These obstacles can be genetic, didactical or epistemological, with the genetic obstacles possessing a psychological origin in terms of the development of the student, the didactical obstacles finding their origin in teaching strategies, and the epistemological obstacles originating in the historical development of the mathematical concept itself. This author explains these obstacles as pieces of knowledge itself.

> The obstacle is of the same nature as knowledge, with objects, relationships, methods of understanding, predictions with evidence, forgotten consequences, unexpected ramifications, etc. It will resist being rejected and, as it must, it will try to adapt itself locally, to modify itself at the least cost, to optimize itself in a reduced field, following a well known process of accommodation... the overcoming of an obstacle demands work of the same kind as applying knowledge, that is to say, repeated interaction, dialectics between the student and the object of her knowledge. (p. 85)

It is also very important to understand that this theory claims that

> An obstacle is ... made apparent by errors, but these errors are not due to chance. Fleeting, erratic, they are reproducible, persistent ... What happens is that they do not completely disappear all at once; they resist, they persist, then they reappear, and manifest themselves long after the subject has rejected the defective model from her conscious cognitive system. (p. 84)

This study takes what already has been identified in the literature, as well as what is identified in the research itself, to see if there are recognizable cognitive obstacles in the process and concept stages of learning to work with integrals. It is important to mention that the theory of cognitive obstacles and, in particular, epistemological obstacles, is a historical theory; this means that when we identify these obstacles in the teaching and learning of a particular subject, we must look to the historical development itself of the subject, to find explanations and reasons behind its existence and persistence. In the case of the integral, if we go to its origin with Archimedes, or afterwards with Cavalieri and Wallis, we can detect that the integral was thought of as a way to calculate not only areas, but volumes and arc length as well, from the very beginning. When it is taught today, the majority of programs, based on standard texts, introduce the meaning of definite integral as synonymous with finding the area under the curve (Czarnocha & Prabhu, 2001). In particular, the calculus course in which the research was done has, as a requisite, the use of the TI-83. In this case, when the student sets up the integral which will represent a volume, surface area, or arc length, and uses the "integrate" feature when graphing, the area under the curve will be shaded.

Tall (2005) coins the term *met-before* to name "a previous construction that is recalled to address a current situation". Met-befores can have the same effects as cognitive obstacles, as there are ideas that were "met-before", and made sense in a particular context, but are now causing problems. An example related to basic mathematics would be subtraction where, in the beginning, the met-before is in terms of not being able to take away more than what is there to begin with, because something less than zero "is not possible". When it is necessary to expand the context, and take into account negative numbers (which can represent temperature, debt, etc.), the met-before no longer applies, but can be an obstacle to mastering the new concept.

Mathematical Fluency

The central concept of this study, *mathematical fluency,* will now be introduced. The concept of mathematical fluency that will be used in this study has its roots in other works, some of them having very different origins. An overview of research and references to fluency will be given, and certain aspects related to the concept of fluency in foreign language learning, considered relevant to the development of the concept of mathematical fluency in this study, will be discussed.

One designation of fluency has been developed by a group of researchers and educators, and is framed in behavioral science theory (Binder, 2003; Bateman, Binder, Haughton, 2002; Layng & Johnson, 1992). The actual applications and results of this orientation will not be analyzed, as they fall out of the scope of this study. Also, their definition of fluency is on another level from what will be used in the present work, as fluency is defined as "the rate of accurate performance and is typically measured as the number of correct and incorrect responses per minute" and "fluency goes beyond mere accuracy to include the pace, or speed of performance". (Brethower, Bucklin & Dickinson, 2000; Bateman,Binder & Haughton, 2003) Speed itself is not a consideration in the present study. However, the motivation that is expressed in many of their articles is very similar to the motivation behind the definition of mathematical fluency that has been developed in this study. For example, Binder (2003) asks "what does it mean to be good at something? How, in other words, do we define competence or mastery?" "We all know fluency when we see it in a foreign language speaker. We say 'she spoke fluent Italian', when we observe a person speaking Italian smoothly, quickly, and without hesitation". (Bateman, Binder & Haughton, 2003, p. 1). Also, these authors mention (p.2) that

> It's no wonder that students in many educational programs often fail to achieve fluency. Instead, they progress by building one non-fluent skill on top of another until the whole skill set becomes "top heavy" and falls apart... For most people math at some point became too unpleasant to pursue further because its foundation contained too many skills that were not fluent and were therefore difficult to apply.

Bateman, Binder & Haughton mention that "fluent performance leads to, among other benefits, longer retention, increased endurance, resistance to distraction, greater ability to apply skills, and faster acquisition of higher-level skills". (p.2). The results that are aimed at by these researchers and practitioners coincide with what any educator desires to achieve. Their methods and theoretical framework are different from those used in the present study.

The NCTM Standards (2000) define fluency as the possession of methods that are "efficient, accurate and general." (p.32). The document mentions "fluency in arithmetic computation" (p.32) and that "Equally essential is computational fluency - having and using efficient and accurate methods for computing". (p. 32) It is mentioned that "Fluency might be manifested in using a combination of mental strategies and jottings on paper or using an algorithm with paper and pencil, particularly when the numbers are large to produce accurate results quickly". (p.32) The document mentions that

> students must become fluent in arithmetic computation – they must have efficient and accurate methods that are supported by an understanding of numbers and operations. "Standard" algorithms for arithmetic computation are one means of achieving this fluency. (p. 35)

In the NCTM Standards, when talking about "computational fluency" it is stated that "Computational fluency should develop in tandem with the understanding of the role and meaning of arithmetic operations" and "understanding without fluency can inhibit the problem-solving process" (p.34). That is, fluency is seen as the possession of efficient and accurate methods, but not necessarily linked with comprehension itself, as it is in the concept of mathematical fluency used in this study.

The *Investigations* project developed at TERC (associated with MIT and funded by the National Science Foundation) has been centered on the idea of fluency. In this context "fluency...involves an interplay of three factors: efficiency, accuracy and flexibility." (Mokros & Russell, 2000). These authors explain that

> Efficient strategies are ones that can easily be applied to given types of problems after an examination ... they are not invented anew each time a problem is presented...Accuracy depends on several aspects of the problem-solving process, among them knowledge of number facts and understanding of important number relationships (they are talking about elementary school learning)...Flexibility requires the knowledge of more than one approach to solving a class of problems... To math educators, striving for mathematical fluency means that 'scope and sequence' documents and Mathematics Frameworks must focus on the complexities involved in learning a mathematical idea." (p.3).

In a MIT publication (2003) the principal researchers of a project aimed to "increase student fluency in mathematics" said

> Mathematics is the language of science and engineering. Our goal is to help our students become fluent in it. We want them to know how to frame questions mathematically and to recognize when and how to

apply mathematical skills and techniques to the problems they face at MIT and in their subsequent careers.

The article also mentions that "The cornerstone of science and engineering education... - Calculus, Differential Equations and Linear Algebra -...are almost as basic at MIT as knowledge of the English language".

Some of the previous ideas do come into play when mathematical fluency is defined in this study, although the definition developed in the present work by no means gives speed a privileged priority, and techniques to achieve fluency are not being proposed. To begin with, the definition used in this study was motivated by fluency as a goal when learning a foreign language. There are learners of a foreign language that have memorized a great number of rules, obtained vast knowledge of grammar, literature, more so than the majority of native speakers, and at the same time, when speaking, commit errors that no native speaker would commit (in English, confusing "he" and "she" for example). It is still not, and might never be, *their* language.

In this study, mathematical fluency will be defined using the three components of efficiency, accuracy and flexibility. Efficiency will be understood as having developed schemas related to the different mathematical areas, which enables the development of strategies, which translate into less time and less effort when confronting problems in these areas. Accuracy will be understood, in the numerical context, as arriving at the correct answer; in the symbolic sense as the correct usage, manipulation and interpretation of mathematical symbols, according to standard mathematical norms, and in the spatial sense as the ability to correctly perform anything from basic geometric operations (such as rotations, reflections) to more sophisticated spatial operations, and to distinguish, identify and situate objects in a spatial context, according to standard mathematical norms. The formal axiomatic aspect is not relevant to this particular study, but can be incorporated when doing studies related to the learning of abstract algebra, analysis, etc. Flexibility is understood as the ability to recognize when a strategy is not working and is not applicable, and to change the strategy at that time in the process. Flexibility requires conceptual understanding and the concept of mathematical fluency, as defined in this study, cannot be divorced from conceptual understanding. To measure these three components, the parameters that measure fluency in foreign language learning are used. They are: speaking, writing, and understanding in terms of reading comprehension and listening comprehension. It is important to emphasize that the four parameters themselves are measures of mathematical fluency. For example, "speaking" mathematics is not considered a method to foment mathematical fluency, but an indicator itself of mathematical fluency. The elements of the five central components of the theoretical framework: duality process-concept; schemas, mental models and strategies; four stage model of mathematical learning; metaphors; and cognitive obstacles, give the language and concepts necessary to be able to analyze mathematical fluency (efficiency, accuracy and flexibility) as measured by the parameters of foreign language learning. This will be further explained in the chapter on methodology.

The four parameters (speaking, writing, reading comprehension and listening comprehension) are used to measure what is called *local fluency* in mathematics in this study. Local fluency is situated in a specific mathematical area (basic math, basic algebra, geometry, etc.); m*athematical fluency* is an amalgam of local fluencies, and is global. At the same time, it is important to mention that local fluencies are not disjoint, and that there are intersections which also have a rich content; what is considered a local fluency in a particular context (for example, differentiation), is itself an amalgam of local fluencies. Mathematical fluency is also context oriented, as mathematical fluency at the calculus II level can be considered an amalgam of local fluencies (in algebra, geometry, differentiation), but it is not expected that the student, even when considered mathematically fluent at this level, should be knowledgeable in, say, linear algebra.

In an analogical sense, through the student's use of the language (by measuring speaking, writing, reading and listening comprehension in the mathematical context, as will be explained in the methodology presented in Chapter V) it will be detected if the student "speaks" (and, that way, understands) mathematics as a foreign language, or if the student has reached a level of mastery in which his expression is that of the "native" speaker (with all due differences in the analogy, as nobody's "first" language is mathematics). The phenomenon of fluency often is a "spiraling" process, as any foreign language learner can testify, and this is true for mathematics as well.

Mathematical fluency, as is detected in the context of this study, has the four components of foreign language learning: speaking (with coherence and logic, that is, correctly, according to standardized norms), listening and reading with understanding (comprehension), and writing. The idea that spoken mathematics can shed much light on students' thought processes is not new at all. Geuther (1986), when analyzing calculus learning, cites Peterson and Swing (1986) as affirming that there are results that suggest that student reports of their thought processes may be better predictors of student achievement and attitudes than observations of student behavior. She also cites Bauersfeld (1979) as saying that it is only in communication with others that a student can test the appropriateness of his/her constructs and correct his/her concepts. In the same vein, Whitin and Whitin (2000), in their book *Math is Language Too,* which is a joint publication of the National Council of Teachers of Mathematics and the National Council of Teachers of English, address the issue of talking and writing in the mathematics classroom. Building upon certain ideas of Vygotsky, the authors explain that "talking does not merely reflect thought but it generates new thoughts and new ways to think". (p. 3) In his studies on memory and thinking, Vygotsky explains that his motivation was brought about "In the light of what my collaborators and I had learned about the functions of speech in reorganizing perception and creating new relations among psychological functions." (1978, p. 38) Other researchers have also observed that "Too often, math time is filled either with teacher explanation or silent written practice" (Curtis & Lovitt, 1968). As will be explained in the methodology section of this study, "spoken mathematics" is analyzed as the subjects are dealing with specific instances of integral calculus, and the responses to specific questions are catalogued according to the relation between successful or unsuccessful performance as determined by the problems the students will work with (which will be transmitted by a combination of spoken and written instructions), and the spoken and written mathematics of the subjects.

The parameters of mathematical fluency, while not specified as such, are implicit in APOS theory (Asiala, Brown, DeVries, Dubinsky, Mathews, Thomas, 1996). Some of the important aspects of their theoretical and instructional components, such as cooperative learning, emphasis on discussion, and the writing of computer programs to foster the action-process-object-schema development can be understood in terms of mathematical fluency, with its local fluencies. In the conclusions of this study, it will be seen how the concept of mathematical fluency can be used both as methodology and as a tool of analysis for many of the current research and discussion groups in mathematics education.

R. Lesh (2000) has developed the concept of *representational fluency.* This concept refers to the fact that mathematical ideas have multiple representations, such as graphs, real-life applications, written symbols and tables. Studies (Gil delos Santos & Thomas, 2003, Cifarelli, V. 1998, Harel & Lesh, 2003) have shown that successful students feel comfortable working with different representations of a mathematical concept, and not just clinging to one of them. Representational fluency is the ability to go back and forth, as well as to manage simultaneity when working with different representations of a mathematical idea. The concept of representational fluency is implicit in the wider concept of mathematical fluency, and really finds its place in the flexibility component.

There has also been some research done on the school level, concerning problem solving through reading and writing (Albert 2000, Baisch 1990, Bickmore-Brand 1993, MacGregor 1993). In particular, MacGregor has pointed to writing as means of "(engaging) students in a process that requires higher-level thinking and active construction of the meaning of (a) concept" (Albert 2005). Mayer (1998), when referring to cognitive, metacognitive and motivational aspects of problems solving, proposed strategies for reading comprehension, writing, and mathematics that, he found, led to student success in problem solving. On the other hand, Smith and Schumacher (2005) found a significant correlation between high verbal SAT scores with regard to the general population tested, and actuarial students. They conclude that "when we recognize that many of the actuarial math courses beyond calculus (such as statistics) do require good reading ability in addition to mathematical ability, it becomes more plausible that VSAT might have an effect on math GPA".

Summary

The elements of the theoretical framework provide the language and concepts to analyze mathematical fluency (efficiency, accuracy and flexibility) as measured by the four parameters of foreign language learning: reading comprehension, listening comprehension, speaking and writing. In particular, the concept of "procepts" helps analyze and pinpoint difficulties that the unsuccessful student might show, as the integral symbol is a prototype of the symbol that simultaneously represents a process (integration) and a concept (accumulation). The four-stage model of mathematical learning provides a categorization which is very useful to understand the learning of new concepts, and gives precise terms that can be used to classify students. Mental models, schemas and strategies are concepts that can also be detected by the parameters of mathematical fluency, and can help shed light on students' performance. The use of extra-mathematical and structural metaphors, some of which are helpful and others which are misleading or plainly wrong, are detected in this study. Cognitive obstacles are also studied in the present work.

The concept of mathematical fluency considers and incorporates previous fluency research, but complements and expands it. Mathematical fluency is a broader concept that borrows from foreign language learning by incorporating the four parameters of reading comprehension, listening comprehension, speaking and writing, but is situated within the domain of mathematics and mathematics learning. It should be made clear that the parameters of mathematical fluency are not a means of promoting or developing fluency, but a measurement technique. In the particular context of this study, speaking, for example, is not considered as a means of developing fluency (as in other research and pedagogical contexts), but as an *indicator* of mathematical fluency itself.

In the present study, the procept that amalgamates the process and concept represented by the integration symbol $\int$ is identified through the parameters of mathematical fluency. The students who evidence maturation in terms of this mental structure (the procept itself), will be seen to have demonstrated higher levels of performance and deeper conceptual comprehension. Classification based upon the four stage model of mathematical learning is made, and it will be seen that this classification is concept-specific; that is, a particular student may be at the integration stage in, say, the use of the disk and shell methods to set up integrals that represent volumes of revolution, while being at the analysis stage when working with functions or trigonometric relationships. The study of students' mental models, identified through the use of the four parameters, leads to the analysis of schemas. It will be seen how important schemas are when students are introduced to new concepts so heavily related to previous geometric and algebraic knowledge. When realizing qualitative research in mathematics education, and especially when using interview techniques, students' use of metaphors can be systematized. This can lead to the detection of patterns of usage, as well as to the detection of individual usage of metaphors; such usage can be either helpful or misleading. There are even certain metaphors, both extra-mathematical and structural, which can play a temporary role in understanding, and must be discarded or modified when dealing with future material. This ties right in to the subject of cognitive obstacles, especially the ones defined as didactical, and which usually have been introduced by an educator as a means of facilitating the learning of a particular concept, but which, on the long run, should be discarded when dealing with future concepts. Often these metaphors or "rules" are carried over into mathematical realms in which they are not useful, or are even false, and it is not clear to the student why they work in one context, and not in the new one. The detection of mathematical fluency as defined in this chapter, and how it relates to students' success with applications, as well as to their comprehension of the underlying notions of calculus, is the goal of the present study. The theoretical framework presented in this chapter is intended to set the stage for the actual research. Although Tall and Gray's definition of procepts, as mental structures that amalgamate the processes and concepts inherent in the same mathematical symbol, together with Knisley's four stage model of mathematical learning, Primm's extra-mathematical and structural metaphors, Brousseau's cognitive obstacles, and the general work done on mental models, schemas and strategies, are the basic providers of the language and concepts used to describe mathematical fluency as detected by the parameters of foreign language learning, other concepts such as abstraction and concept image will also be mentioned in the chapters on results and conclusions. For that reason, this chapter's intent was to offer all the elements needed to frame the present study; also included are some elements not explicitly called upon when presenting the results and conclusions, but whose presentation could help in the understanding of the foundations upon which the present study has been built.

CHAPTER III

LITERATURE REVIEW: LEARNING AND CALCULUS

There has been a lot of activity in terms of research and reflection on calculus teaching in the past twenty years, not only in the United States, with the "Calculus Reform", but in other countries as well. The "Calculus Reform" in the United States has produced many studies, the majority dealing with comparative questions between the reform and standard calculus courses. The majority of these studies do not shed much light on the research questions of this study. On the other hand, studies dealing with conceptual and cognitive issues in calculus learning are not so abundant, the majority of them addressing the concepts of limit and derivative, more than issues related to the integral. The research on conceptual questions related to the mathematical object "integral" takes place in the setting of the standard second calculus course, which assumes that the student has studied the Fundamental Theorem of Calculus. That is why this literature review will begin with some research that has been done on students' understanding of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

Several authors (Knisley, 2000; Tucker, 1995; Carlson, Persson, Smith, 2003) have commented that, instead of unleashing an important conceptual insight that links the two basic aspects of calculus, students find, when presented with the Fundamental Theorem of Calculus, that "the amazing connection between differentiation and integration is anti-climactic, at best." The study on the Fundamental Theorem of Calculus most related to the present study is by P. Thompson (1994) in which, through a teaching experiment in a course, especially devised, on computers in teaching mathematics, and geared towards a relatively "sophisticated" group of students (7 senior mathematics majors, 10 masters students in secondary mathematics education, 1 masters student in applied mathematics and 1 senior elementary education major, having completed at least 3 semesters of calculus), he studied students' insights into the Fundamental Theorem of Calculus. The analysis was divided in three parts: "Students' images as expressed during the teaching experiment and their contribution to students' difficulties, issues of notation, and implications of the ... teaching experiment for standard approaches to the Fundamental Theorem and introductory calculus in general." (p. 268). The issue of notation will be discussed below, in the subsection on notation, as the author himself commented that "Their orientation toward notational opacity, while having nothing to do with conceptual difficulties with the Fundamental Theorem of Calculus as such, certainly contributed to their not having grappled with key connections." (p. 270). This last aspect, the "key connections", was a very important finding of this teaching experiment. In the transcriptions of the students' participation in class, as well as the results of the follow up assessment which consisted of "...four items to clarify possible sources of difficulty – two items on interpreting a difference quotient and two items on Riemann sums as function", the lack of connections was evident in the majority of the students in every question. In particular, it was evident that the main idea of the Riemann sum as a function was not understood. On one question, a graph was given as a function $q(x)$ and the students were asked to sketch a graph of $z(x)=\sum_{i=1}^{n} q(i\frac{x}{n})\frac{x}{n}$ for $n=1000$ and $x \in [0,5]$. Only 8 out of the 17 students sketched appropriate graphs. When referring to students' images, the author states that

> Some students had not come to coordinate the variations of upper limit of summation and the variations in the index of the summation; some students had not coordinated the actions of forming a sum and multiplicatively constructing an accrual to a sum. Other students had mastered both of these co-ordinations but could not coordinate that ensemble of actions with the action of comparing multiplicatively the growth in an accrual with growth in one of its constituent quantities... Operational understanding of the Fundamental Theorem entails the coordination of these actions so that the scheme remains in balance. Operational understanding of the Fundamental Theorem allows one to hold simultaneously in relation to one another the mental actions of forming accruals, accumulating accruals, and comparing an accrual to one of its constituent quantities multiplicatively. (p. 270).

As far as the implications for the standard approaches to the Fundamental Theorem, the author questions the typical proof that is given to motivate its use in calculus. He states that "The problem with the typical proof is not so much in the proof as that it is presented as modeling a static situation. It is presented in such a way that nothing is *changing.* If students are to understand that $F'(x)$ is a rate of change, then something must be changing." The conclusion is that "...this teaching experiment suggests that a great deal of image-building regarding accumulation, rate of change, and rate of accumulation must precede their coordination and synthesis into the Fundamental Theorem." (p.272). Although the author himself does not deal with the question of fluency, it is obvious from the reports of the results, and the fragments of student participation, that part of the problem also lies in the building of non-fluent skills on top of other non-fluent skills, as has been mentioned in the literature.

Ferrini-Mundy and Graham (1994) conducted a study on calculus learning and understanding of limits, derivatives and integrals. This study was done through interviews, and the questions related to integration were "...intended to check .... the interpretation of antiderivative tasks but then to move on to (the) understanding of the concept of the definite integral and the Fundamental Theorem of Calculus". (p.41) The question consisted in presenting two integrals, $\int x^2 dx$ and $\int_0^2 x^2 dx$. The interviewee who was used to exemplify the study in this article was very strong in procedural skills, but "demonstrated a great reluctance to use geometric interpretations as a help in completing an algebraic process". She found the difference between the two integrals to be unimportant, that both were " 'taking the antiderivative - but the second is more definite' ". Another interesting expression of this interviewee, when related to the present study, was "when you do the antiderivative of a function, it's the area between the graph of that function and the x-axis ". The authors mention that "this explanation seems to be related to her fundamental understanding of the definite integral". This is an aspect that is also probed in the present study.

There are some studies that suggest different methods and techniques for introducing the Fundamental Theorem of Calculus, but these happen to suppose, on the basis of common consensus more than specific empirical evidence, that there is something wrong with the way it is introduced and dealt with in the standard calculus course. F. Cordero (1989) suggests that "One appreciation of a didactic nature is that the Cauchy-Riemann definition of integration is prematurely presented in the IC (integral calculus) discourse and is even, perhaps, unnecessary." (p. 62) This author feels that the notion of antidifferentiation is what really connects the integral to the derivative, because in this case it is in this context where it is "linked to the differential of a variable or a quantity". He also justifies this position in terms of historicity, alleging that the "taking of a differential" element preceded the Riemann approach, when calculating quantities. To justify this he quotes a 1837 French calculus text which affirms that "A quantity which through differentiation yields a proposed differential is called the integral of such a differential." There have been proposals to change the standard form of presenting concepts in integral calculus based on historicity (Czarnocha & Prabhu, 2004) but, while they can be illustrative and even effective when applied in a teaching situation, it is not clear that there could really be an echo in terms of general implementation. On the other hand, Tall (1986) proposed, as the title of his article indicates, "A Graphical Approach to Integration and the Fundamental Theorem". Although this article dates almost twenty years, it is clear that this author was an advocate of the technology of that time, and justifies the use of numerical methods and graphical insight with a computer, under the grounds that this way it is much more meaningful to talk about a "big" n. He even explains how the exploration of properties of the area function as related to original function can let students preview powerful theorems from analysis. Gordon and Gordon (2003) suggest using data analysis to discover the Fundamental Theorem of Calculus by means of curve fitting.

Carlson, Persson and Smith (2003) claim that a process view of function and a certain level of covariational reasoning (coordination of an image of two varying quantities, while attending to how they change in relation to each other) are necessary prior to students' study of the FTC. Their objective was to "provide additional clarity about the understandings and reasoning abilities involved in learning and using the FTC." (pp. 165, 166). They realized a study with 24 beginning calculus students and used a pre- and post- calculus assessment instrument, based on what they developed and called the "FTC Framework". The of a function's input variable with the accumulation of instantaneous rate-of-change, from some fixed starting value to some questions were designed to detect:

1) notational aspects of accumulation;

2) coordination of a function's input variable with the accumulation of instantaneous rate-of-change, from some fixed starting value to some specified value for various contextualized situations;

3) understanding of the statements and relationships of the FTC.

They also interviewed 4 of the students, who were asked to complete tasks and verbalize their thinking while responding to the written questions. In their conclusions, these authors state that, in spite of a strong understanding of the notational aspects of accumulation and covariational reasoning by the students, and even of the relationships of the FTC when compared to studies realized with other groups of students by other authors on the FTC, (it isn't clear if this is attributed to the structure and content of the particular course, or the particular composition of the students), understanding the statements and relationships of the FTC needs to be strengthened. This leads them to suggest the "development of additional ideas for curricular tasks and prompts to better assist students in developing these understandings and reasoning abilities". (p. 172).

Notation, Symbols and Metaphors

Thompson (1994) claims that students often use notation in an "opaque" manner. He interprets this as the way students have learned to cope and obtain a certain degree of success in their high school and college mathematics courses. In his study on the Fundamental Theorem of Calculus, mentioned above, he stated that more than thinking through and interpreting the notation, it was found that "...students would not interpret the notation with which they worked, but would instead associate patterns of action with various notational configurations and then respond according to internalized patterns of actions." (p. 270). In another study (Barzilai, 1999), mistakes in calculus and precalculus mathematics were analyzed. The study was realized by analyzing journals at several institutions (a large public research university, a medium sized private research university, and a small private liberal arts college). The evidence was that these "notationally related" student mistakes are common and transcend particular institutions; also, "the data suggest(s) that certain conceptual student misunderstandings, relating to the dual nature of functions (operators versus objects), and other dualities (e.g. constants versus variables) underlie many of these notational mistakes" (p.1). The author suggests classifying RCMs (regular calculus mistakes); the worst that could happen would be to have a "canonical list" of such errors, and therefore be able to "train" students to avoid them. Of course, the ideal goal would be to address the conceptual problems which underlie the mistakes.

Tall (1992) mentions that the Leibniz notation $\frac{dy}{dx}$ can cause notorious conceptual problems, as well as the relationship between the dx in this expression and that of $\int f(x)dx$.

Frid (1994) reported upon the results of a study that involved three different approaches to calculus instruction, that is, a technique oriented course, a concepts-first oriented course, and an infinitesimal oriented course. This author found that "the role symbol systems play in learning was ...evidenced." (p. 94). In this study the role of language in mathematics learning was classified and the analysis of the students' language was done through problem responses. This involved counting the occurrences of a student's use of symbols and technical or everyday language and "...it was noted if students merely performed operations with symbols in responding to a differentiation problem or if they introduced operator or first principles notation as a representation of the differentiation process. (p. 73). It was found that the use of physical and visual experience helps the student construct meaningful representations of the important concepts in calculus and this, in turn, enhances the connection to symbolic representation, and makes it more meaningful. The author suggests that instruction be designed from a social constructivist perspective "to promote calculus learning as a process of subjective construction of publicly shared knowledge". (p.93).

Oehrtman (2003) analyzed verbal and written language, referring to limits, in first year calculus students as he identified reasoning processes with metaphors that students use. He found that the "students did not reason about limit concepts using motion metaphors" in spite of the fact that there is a "predominance of motion language used when talking about limits and abundant proclamations that intuitive, dynamic views of functions should help students understand limits." (p. 403). However he did find a significant presence of non standard metaphors that, while technically incorrect, were helpful to the students' understanding as organizers of ideas and tools with which they were able to develop further connections. An example of this type of metaphor is the "collapse" metaphor, in which "students characterized a limiting situation by imagining a physical referent for the changing dependent quantity collapsing along one of its dimensions, yielding an object one or more dimensions smaller". (p. 300). For example, some students explained "Gabriel's horn" (the volume of revolution generated by rotating $f(x)=\frac{1}{x}$, $x\in[1,\infty)$ about the x axis) in the following way. They imagined the three dimensional solid as turning into a line with no volume as the radii of the cross-sectional disks, produced by revolving the curve, got smaller and smaller as x got larger, and tended to infinity. This is just one example of the type of result found. The students were taken from an introductory calculus sequence, and they participated in interviews and submitted writing samples about the limit concept. The categories of the metaphors were refined by coding, and the metaphors were classified using an instrumentalist approach (p. 398).

Arslan, Gamid and Laborde (2002), question "algebraic dominance" in the teaching of differential equations. Through an experiment with students who had

3 years of university with a specialty in mathematics, and who were preparing for the competitive examination for teachers, the researchers analyzed four worksheets. The results show that the tendency of students, even after being exposed to qualitative methods for obtaining results, could not seem to attenuate the impact of the dominance of the algebraic in all their previous experience, and they frequently forced the notation. "The majority of the students have difficulties in giving the names of the dependent and independent variables (and) also in **identifying a differential equation."** (p.7). In some way, the authors seem to include algorithms and procedures in their definition of algebraic; taking this into account, some of the examples are striking. At one point, the authors refer to another study that one of them realized with Mexican students (Laborde & Moreno, 2003), in which having a removable curve in the screen of Cabri, "they had to associate it, WITHOUT INTEGRATING, with one of the differential equations provided in the list." Instead of relying on the geometrical characteristics and invariants that were provided visually, the students tried to integrate the equations, even in cases when the answers were meaningless.

Problem Solving

Problem solving may be one of the most important indicators of transference of knowledge and techniques (strategies), within the same domain and between domains. The literature on problem solving and integral calculus is, almost exclusively, related to pedagogical approaches, in particular reform versus standard, but not analysis of mental models, metacognition, abstraction, transference, etc. However, there are some studies related to calculus problem solving in general. Cifarelli (1993) realized a study in which he wanted to follow the representation processes in the problem solving situation. The subjects were from freshmen calculus courses at the University of San Diego. The author mentioned that "...the cognitive studies that have been undertaken have seldom focused on the ways that learners actively modify their problem representations when they encounter problematic situations." The idea of this study was to see how problem posing itself is implicit in the process of problem solving. Cifarelli also mentions that "Research suggests that the success of capable problem solvers may be due ... to their ability to construct appropriate problem representations in problem solving situations to use as aids for understanding the information and relationships of the situation at hand." This is an approach that differs from that used in studies cited by Cifarelli, where "a solver's ability to recognize similarity across tasks that embody similar 'problem structures' is taken as evidence that the solver has developed an appropriate problem representation." That is why this study in particular was designed to "...acquire an understanding of the processes used by learners to construct and/or modify problem representations in problem solving situations." (pp. 3, 4). The study was realized through interviews, in which it was possible to observe the subjects experiencing "dilemmas", which, if things went well, provided them with the opportunity to solve these dilemmas and, in the process, further their conceptual knowledge. Of course, by "solving a dilemma", the student was making progress in the problem solving process by abstraction in a domain specific context. At the same time, the chaining process occurs, because once the problem solver modifies the problem representation, the hierarchies of mental objects upon which he is working, as well as the meanings (correct or not) are transformed (for example, once the function for an optimization problem is created, it becomes the "object"; it also turns from "signifier" into "signified" as it is operated upon). Another study which deals with calculus and problem solving is the article "Even Good Calculus Students Can't Solve Nonroutine Problems" (Mason, Selden & Selden, 1994). In this study, part of a series that the authors realized, a group of A and B calculus students who had taken first semester calculus the semester before the study, were given a set of nonroutine problems to solve, as well as a set of routine problems that tested their general knowledge and memory of what they had seen in their calculus class. They were compared with a group of C students who had been given the same routine and nonroutine problems in a previous study and who had been unable to solve problems for which they had not been taught a method of solution, even though they had sufficient knowledge to solve the problems. The A and B students did not fare much better, as "two thirds of the students could not solve a single problem correctly.". (p. 25) It was interesting to see that, according to the authors, students who monitored their work did somewhat better; even though it wasn't pointed out in the article, this is very related to the findings of Cifarelli, in terms of the importance of representation and mental models within the solving process of a particular problem, not just in terms of generalization across problems. This also gives evidence of the presence and importance of chaining. It is also a consensus that the accumulation of more factual knowledge will not necessarily lead to this nonroutine problem solving ability. Another important aspect to take into account from the results of this study (and related to the study by Arslan, Chaachoua and Laborde mentioned in the subsection on notation), is that the students tended to look for the solution to calculus problems in arithmetic and algebraic techniques more than in the calculus knowledge that they knew they were being tested on. The authors raise the question of whether it takes time to get comfortable with a new subject, and perhaps the students at the end of the two year sequence (calculus and differential equations) would be more inclined to make use of calculus.

It has been noted that

> Because students are used to listening, taking notes and learning procedures to solve standard problems, it is crucial that the teacher renegotiate the 'didactic contract' in order to set up a 'mathematical community' in which students propose and evaluate conjectures for themselves using solid mathematical reasons. (Selden & Selden, 1997).

These authors cite the work of Schoenfeld, who combines traditional and non-traditional techniques. It is emphasized that, in problem solving, it is important for the student to express his ideas verbally, and to get feedback; this, in itself, is very related to our construct of mathematical fluency and the four parameters. Schoenberg (2000) gives an example which, he declares, "The issue concerns 'metacognitive behavior'...specifically, the effective use of one's resources (including time) during problem solving". In this case, while studying integration techniques, the students were given, to solve, the integral $\int \frac{x}{x^2-9}dx$. It was expected that the students would use substitution $(u = x^2 - 9)$ to solve the problem quickly (efficiently!). While half the class did proceed that way, others used partial fractions or trigonometric substitution; these methods, when correctly used, lead to the proper results, but the students lost so much time that they performed, in general terms, much worse in the exam itself, than those that used substitution! The idea that came out of this example is related to the notion of "strategic choices" during problem solving.

Other Cognitive Aspects of Calculus Learning

Two concepts of precalculus and calculus learning that have been identified as complex and problematic for students, and studied in depth during the last twenty years are *functions* and *limits* (Tall & Schwarzenberger, 1978, Sierpinska, 1985, Davis & Vinner, 1986, Williams, 1991, Dubinsky, 1992, Gray & Tall, 1992i, Ferrini-Mundy & Graham, 1994, Czarnoch, Loch, Prabhu & Vidakovic, 2001). Some of this research, by means of qualitative techniques and systematization of responses, has discovered patterns that reveal "standard" misconceptions that students possess. It has been seen that even students entering into a college calculus sequence often have a very primitive notion of a function and often reject discontinuous or piecewise functions as functions. It has also been detected that students expect an algebraic representation when manipulating functions, and have difficulty in interpreting graphical representations (Heid, 1988).

Limits present a series of problems for the student, and the different ways that they are introduced within the calculus curriculum will influence the way they are understood and the misconceptions that come about when they are used later on. Sierpinska (1985) has studied epistemological obstacles related to limits. In her studies she has focused upon students' attitudes, intuitions and problems related to the notion of infinity. In general terms, the studies (Schwarzenberger & Tall, 1978, Davis and Vinner, 1986) show that students think that a formal definition of limit, $\lim s_n = l$, implies that the sequence gets close to l, but never coincides. On the other hand, (Ferrini-Mundy & Graham, 1994) when a function is continuous, students often think that finding the limit is no more than an evaluation, and that the notation $\lim_{x \to a} f(x) = L$, and the "proof" with the $\varepsilon - \partial$ definition is meaningless. As the definite integral is usually introduced in the standard first calculus course curriculum by means of Riemann sums, if the notions of limits in general, as well as limits of partial sums, are not clear, the whole explanation of the integral as the sum of n-partitions, as n tends to infinity, does not seem to justify why the integration process gives an exact answer, instead of an approximation (Czarnocha, Loch, Prabhu & Vidakovic, p.1).

Ferrini-Mundy and Graham (1994) conducted a study on calculus learning and the understanding of limits, derivatives and integrals. Six first semester calculus students were selected and interviewed over a two semester period. The researchers "provide(d) tasks that help reveal students' ways of thinking about the central concepts of functions, limit, derivative, and integral" (p.33). One very interesting result of their study, related to the Fundamental Theorem of Calculus, is presented in the first section of this chapter. Another aspect, related to the present study, is seen in their general observations:

> Calculus students will actively formulate their own theories, build their own connections, and readily construct meaning for problem situations. These processes seem to be influenced strongly by previous experience and knowledge. There are powerful tendencies to call upon familiar examples and frequently-used patterns. (p.43)

In other words, previous knowledge is what provides the content of the analogies and allegories as the student passes through the stages of the mathematical learning process. They also concluded, on the basis of their study and others, that "there is substantial evidence...that students in secondary school and beyond have limited facility in making connections between representations in the Cartesian plane and representations in the formal algebraic system." (p. 44) This is, of course, a tremendous obstacle in calculus learning, especially when dealing with the type of applications of the integral that are studied in the present work.

Another conclusion of theses authors, relevant to the present study, that there is "...solid evidence that students can perform the procedural tasks of calculus with rather astonishing success while displaying conceptual understandings that are not what we would like to have in place." (p. 44) This is related to the flexibility aspect of mathematical fluency as designed in the present study, with its component of conceptual understanding.

Summary

While calculus teaching and learning is one of the fundamental concerns of collegiate mathematics education research, the bulk of the studies in this area are related to measuring the results of reform versus traditional calculus, and the implementation of specific techniques, more than cognitive aspects. On the other hand, studies on the cognitive aspects of calculus have concentrated more on precalculus requisites, such as functions and variables, as well as the standard first calculus course concepts such as limits, and differential calculus in general, than on integral calculus. The present study hopes to offer some results on specific aspects of integral calculus, that is, applications of the integral, using the construct of mathematical fluency.

CHAPTER IV

CONTEXT AND METHODOLOGY

Context

The preliminary study, as well as the main study, was realized with a standard second calculus class at a public community college in the Northeastern United states in which Larson's *Calculus of a Single Variable* (2003) was the text used. Each year over 15,000 individuals study at this particular community college either full- or part-time and, while the student body is diverse in terms of educational objectives, the student who is taking calculus II is usually planning to transfer to a four year institution, and major in engineering or science. The qualitative research paradigm (Ernest, 1988) provided the methodology "that is, a general theoretical perspective on knowledge and research, that (allowed) specific methods, instruments and techniques to be selected for (this) particular project". (pp. 22 & 23). The preliminary study, realized the summer of 2005, was a combination of *action research* and *interviews* or *case studies* (Romberg, 1992). Action research is research in the context of efforts to improve performance, such as the classroom. During the spring semester of 2005 I was the professor of a small calculus II course (7 students). I could follow the students' progress by analyzing exams, quizzes, assignments and class participation; once the semester was over, I interviewed three students, an "A" student, a "B" student and a "C" student. These interviews were tape recorded, transcribed, and analyzed according to the criteria that I will explain in the sub-section on methodology. The main study was realized during the fall semester of 2005, and consisted of 3 interviews each, with four students, also audiotaped, in the same setting of a standard calculus II course, this time with 20 students, and with Larson's text. I team taught this class, and my role as a researcher was two-fold. On the one hand, I did action research as I had a direct part in the class learning process as well as the course evaluation and its design. On the other hand, I was also an observer part of the time in the class. The analysis was based strongly on the interviews, although I took into account comparisons with the general performance of the class, which meant class interaction as well as written assessments. The interviews were designed to assess applications of the integral and a general notion of The Fundamental Theorem of Calculus.

Methodology

The analysis of the interview questions used to evaluate performance and fluency are framed in the theoretical aspects of process-concept duality; schemas, mental models and strategies; the four stage model of mathematical learning; metaphors; and cognitive obstacles. These theoretical aspects are seen as providing the language of analysis of mathematical fluency when dealing with *applications* (such as setting up integrals to generate solids and surface areas of revolution, arc length, and work), and *definite and indefinite integration* (which shows how the applications connect to the underlying notions of accumulation and infinite sum). It is very important to mention that this study does not deal with the actual concept of infinity, which in itself is a fundamental area of research.

As was mentioned, five components of the theoretical framework provide the language and concepts which permit the detection of mathematical fluency; *mathematical fluency*, in turn, consists of three components: efficiency, accuracy and flexibility. To measure fluency, the four *parameters* of foreign language learning are used, parameters being understood as "characteristics" (The Oxford American Desk Dictionary and Thesaurus, 2001) or reference points that define something. These parameters are: speaking, writing, as well as listening and reading comprehension.

Although the present study was not formally based on the APOS framework (Asiala, M., Brown, A., DeVries, D.J., Dubinsky, E., Mathews, D., & Thomas, K., 1996), the "...structured set of mental constructions (know as) *genetic descomposition*" (Loch, Prabhu & Vidakovic, 2001), developed under the APOS guidelines is a reference point for the methodology of this study. The concept of genetic decomposition comes into play when conducting qualitative research. Following this methodology, one hypothesizes about the "mental constructions that a student might make when learning a specific mathematical concept." The framework privileges the "researcher's own understanding of mathematics along with her or his learning and teaching experiences." (p.3) It is this last aspect, the researcher's own understanding of mathematics, along with his or her's learning and teaching experiences, that makes research in mathematics education, along these guidelines, so *sui generis*. In itself, the concept of genetic decomposition refers to mental models and schemas, but the actual research, under this methodology, must be done by a working mathematics educator who has been trained with a strong mathematical background.

The research questions address the learning of new concepts introduced in the second calculus course, and how this learning manifests itself, and is different, in the fluent and non-fluent students. Performance refers to level of success in answering the questions that will be displayed below (as well as classroom participation and exams), and the classification of "fluent" and "non-fluent" refers to efficiency, accuracy and flexibility, as measured by the four parameters, when solving the problems. The research questions are presented in the introductory Chapter I, and again in Chapter VI, Conclusions.

Measurement Techniques for Mathematical Fluency

The following are the measurement techniques for mathematical fluency:

I. Comprehension of written mathematics (*Reading Comprehension*).

Students were given written problems and asked to:

i. Read silently;

ii. Read aloud;

iii. Explain what they understood.

II. Criteria for *Writing*;

An analysis of students' written work was made to observe:

i. Logical structure, as represented by the sequence and order of steps in attacking a problem;

ii. Strategies of attacking a problem;

iii. Sketches and graphical representations;

III. Criteria for *Speaking:*

Students' verbal expression was coded to detect coherence and precision in terms of:

i. Extra-mathematical and structural metaphors;

ii. Terms related to the subject;

iii. Descriptions of strategies;

iv. Descriptions of procedures;

v. Descriptions of concepts.

IV. Criteria for comprehension of spoken mathematics (*Listening Comprehension*)

I explained in mathematical terms (at the mathematical fluency level of the calculus II course) some process or concept, and I asked the students' what they understood. When students were proceeding in a mistaken manner, I would intervene, and take note of their response to my orientation.

Before continuing with the description of the interview techniques and questions, it will be useful to exemplify how the criteria for the measurement of mathematical fluency were employed. In terms of reading comprehension, the students were confronted with an interview question, and were told to read it for themselves, then to read it aloud, and then to explain what they understood, as well as what result the written instructions specifically asked them to arrive at. They were then told to begin to work, and to verbalize their ideas and procedures as they were writing and sketching. It was here that the logical structure of their thinking process, indicated by writing and speaking could be detected. For example, when asked to set up an integral that would be used to find the volume of a solid of revolution, the logical sequence would be to first find the region bounded by the given equations, look for the intersection points, identify the line about which the region would be revolved, as opposed to immediately deciding to use a particular method, such as washer or shell. Listening comprehension was evaluated when, for example, I would indicate in some manner (sometimes just by reiterating the actual question or instruction) that there was an error in the procedure they were following. The less fluent students, as will be seen, often did not respond to my observation, persevering in their errors, or did not understand the orientation or hints I would give. The students with listening comprehension would capture their errors and change strategies, showing that they had understood my verbal intervention. Lack of fluency as illustrated by the listening comprehension parameter could be seen when instructions or explanations were given verbally, and the response of the student, whether written or verbal, showed that they had not understood. It is important to mention that listening comprehension is assumed when teaching is done in the lecture format, this being the most usual form of mathematics teaching at the college level up until this date.

The Interviews

Motivation

The fundamental question, from the perspective of calculus learning, that underlies this study is how the calculus student conceives the integral, in the different contexts in which it appears. The integral, in a usual second course is approached as something already familiar to the student, having been presented in the first calculus course in a highly non trivial manner. The student supposedly has understood and is capable of using the concepts of antiderivative, Riemann sums and the Fundamental Theorem of Calculus, as well as performing the integration process at least by means of the power rule and substitution. It is usual that, near the beginning of the course, the student will be exposed to "applications of the integral", in which the initial idea of "calculating the area under the curve" will be extended to the calculation of volumes of solids of revolution by the disk, washer and shell methods, as well as the calculation of surface areas generated by revolution of a curve around a line, arc length, work and moments. The question is if, at this point in time, the student is asked to perform the multiple operations that he has been exposed to and then explain why the integral, as a process, can be used to calculate areas, volumes and one-dimensional lengths, will the original concept of the integral as accumulation, represented by a sum, be called upon as a natural explanation, or will the question itself appear meaningless or "hard" to answer.

For example, when the process of calculating volumes and surface areas of solids of revolution is introduced as an “application” of the integral, how is this reconciled with the original motivation of the integral as a method of calculating the area under a curve. Is it usually even addressed in the syllabus? At what moment is the integral itself a concept? At the same time, the very first motivation of the integral is usually the antiderivative. When the integral sign is first presented as the equation: $\int f(x)dx = F(x)+C$,

is it clear for the student that he is being asked to operate on a function (as an object, like a number) and give as an answer a new family of functions, which is the antiderivative? Then what happens when it is said that this same equation can be presented as:

$\int F'(x)dx = F(x)+C,$ or even

$$\frac{d}{dx}[\int_a^x f(t)dt] = f(x)$$

These are the types of questions that were in mind when deciding upon the actual questions to be included in the interviews.

The wording of the questions was transcribed verbatim from a standard calculus text (either Larson or Stuart), as it was important to use the language that the students were confronted with when asked to perform the homework assignments. The first two of the three interviews were based on the generation of solids of revolution using the disk, washer and shell methods. The emphasis was not on actually solving the integrals but, in the first question, on setting up the integrals and choosing the method and, in the second question, on identifying the solid, the original region and the method used in integrals that were already set up. In the second question, the student is also asked to sketch what the integral would look like if it was to represent the area under the curve. The idea behind this was to indirectly check if understanding of the definite integral, in the original presentation, had been achieved. The third interview involved the third and fourth questions, the third being related to arc length and the fourth to the Fundamental Theorem of Calculus.

Interview Questions

1) Set up the integrals that would permit you to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated lines.

a) $y = x, y = 4x - x^2$; about x-axis.
b) $y = x^2, x = y^2$; about $x = -1$.
c) $y = \sin x, y = 0, x = 2\pi, x = 3\pi$; about the y-axis.

2) Sketch what the following integrals should look like. Why should we use the integral to calculate the volume of solids of revolution?

a) $V = \pi \int_0^4 (\sqrt{x})^2\, dx$

Sketch what this integral would look like if it represented the area under the curve.

b) $V = \pi \int_0^2 [16 - (y^2)^2] dy$

c) $V = 2\pi \int_0^1 x^2 dx$ and $A = 2\pi \int_0^1 x^2 dx$

3) How do you determine arc length by using an integral? Can you explain, and/or draw a picture, showing how the formula $s=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$ was developed?

4) Why do we use the integral to find arc lengths, areas and volumes?

5) The derivative can represent a rate of change. If a function represents a rate of change, it is the derivative of another function. Let f' represent the rate of change function.

What does $g(x)=\int_a^x f'(t)dt$ represent?

What does $\int_a^b f'(x)dx$ represent?

What is the difference between the two integrals?

Method of Data Analysis

The method of analysis was based on the measurement techniques for mathematical fluency. The interviews were tape recorded, and transcribed after each interview. Although the four parameters of mathematical fluency, reading comprehension, listening comprehension, speaking and writing, are listed as separate and there are different techniques for detecting mathematical fluency through each one of them, at the time of the interview there was no clear separation. For example, the spoken mathematics (speaking), which were analyzed according to the expression of extra-mathematical and structural metaphors, descriptions of strategies, terms related to the subject and descriptions of concepts, often came about as an explanation of what the students understood from the written problems (reading comprehension). The analysis of written mathematics (writing) could not be divorced from their understanding through listening and reading comprehension.

Once the transcribed material for each student was separated into four groups, according to the four parameters (although there were overlaps, in the sense that the same material was sent to more than one of the parameters; for example, the use of a metaphor could be in the "speaking" category, but its initial motivation could come from the comprehension of written mathematics, that is, reading comprehension) each group was examined for efficiency, accuracy and flexibility, the criteria for mathematical fluency. In determining, say, efficiency, the language and concepts from the theoretical framework were used. In particular, the procept construct, mental models (together with schemas and strategies), the four-stage model of learning of a new concept, extra-mathematical and structural metaphors, and cognitive obstacles provided the bulk of the terminology and concepts used to analyze mathematical fluency, as measured by each one of the four parameters. The presentation of results in chapter V is realized according to this scheme.

Data Sources

The action research took place in the classroom, where I was one of the two professors responsible for the class. The syllabus and the relevant exam are included in Appendices B and C. At the same time, I was an observer during the class sessions when I did not teach (we usually alternated sections), and could analyze the dynamic in the classroom, in which all the students were present. The criteria and techniques of mathematical fluency, as described above, together with the language and concepts of the theoretical framework, were what guided these observations, and the conclusions that were reached in function of them. Case studies were done by interviews and in particular cases during the interview process, when certain tendencies were suspected to exist, it was possible to test in the context of the classroom, to see how the non-interviewees performed or showed their understanding when confronted with similar situations. This was usually done through class exercises, actual questions to specific students or the entire class, and analysis of homework problems and the exam. Sketches and notes from students' participation in the interviews are included in Appendix D.

The Interviews

Four students were interviewed over a time period of three weeks, one interview a week, to sum a total of twelve interviews. The interviews were audiotaped and then transcribed. The students' names were changed to protect privacy. All the students in the calculus II class who participated in the study are planning on transferring to colleges or universities.

Key Participants

Susan. Susan was a sophomore, who was planning to transfer to a large state university and pursue a career in architecture. She didn't need to take calculus II to transfer, but wanted to take as many math courses as she could. She had been my student in the second precalculus course (advanced algebra and trigonometry) a year before (fall 2004), had taken calculus I in the spring of 2005 and withdrawn, and completed calculus I during the summer of 2005 with the professor with whom I was team teaching. She received an A in precalculus, and a B+ in her calculus I course. She was a very hard worker, and would come to office hours and look for extra help from anyone in the math department on an almost daily basis. As will be seen in the interviews, her geometric intuition and spatial reasoning in general were good, but her algebra and trigonometry skills were very poor, even though she had received an A in the advanced algebra and trigonometry course the year before. She seemed to learn for a context specific subject and the exams, but did not make connections or store that knowledge in her long term memory.

Tom. Tom was a sophomore who was planning to transfer to a state university to study sound recording. He had been accepted in the music department of the university but, due to his grade point average, he had not been accepted to the university, and for this reason was taking courses at the community college. He had been my student the previous semester in calculus II, and had not finished the course. His attendance had been erratic, and his study habits poor, but when I had worked with him, I found that his algebra skills were excellent, as was his comprehension of new concepts. The semester of the study he had been attending regularly and his performance was very good. He had received a B in calculus I with the professor with whom I was team teaching.

Laura. Laura was a sophomore who was planning to transfer to a state university, and then continue studying veterinary medicine at a prestigious private university. She did not need calculus II, but felt that she would be more competitive if she took more math courses. She had taken calculus I with the professor with whom I was team teaching, and had received an A-. She had also been hired as a peer tutor for lower level courses. As will be seen in the interviews, her geometric and spatial reasoning were very poor, and she openly said that she wanted rules and procedures, not drawings. She also commented, when asked about the applications, that "I hate applications. I just want to do derivatives, integrals, I don't care what they mean. I don't care if it's an area or a volume, I just want to integrate".

Paul. Paul was a sophomore, and planned on transferring to a state university to study electrical engineering. He had also been in my calculus II course the previous semester, and had to withdraw for personal reasons. He had been a very responsible and hardworking student, and showed these same characteristics in the fall 2005 course. He was very proficient with the calculator (a TI-89, although the course syllabus called for the TI-83), and would always use it to check work done during the lectures. His precalculus skills were good, and he showed interest in understanding the underlying conceptual material, as will be seen in the interview transcriptions.

Summary

As I developed my methodology, conducted my preliminary study, and analyzed data, my research questions, focus and emphasis underwent substantial modifications. The concept of mathematical fluency went from being an intuitive tool that was helping me structure the study, to an actual measurable theoretical construct. This in itself is reminiscent of the Constant Comparative Method (Glaser & Strauss, 1967). The use of *case studies* through *interviews,* within the context of *action research,* are the specific methods and techniques that stem from the qualitative research paradigm (Ernest, 1988), which has proved to be invaluable to the development of methodologies in mathematics education research. The combined role of professor, which permitted the logistics of the action research, with observer and interviewer, made a privileged situation in which to realize the research. I would never have been able to detect the nuances of mathematical fluency if I had not engaged in qualitative research. However, this having been accomplished, it will be explained, in the chapter on conclusions and recommendations, how to envision the measurement of mathematical fluency on a large scale, using quantitative data as well.

CHAPTER V

RESULTS

The results presented in this chapter are based on interviews, classroom observations and activities carried out during the fall semester of 2005, at a community college in the Northeastern United States, where I was one of the instructors assigned to the Calculus II class, which was team taught. In particular, the study was focused on the material related to applications of the integral in terms of generating and calculating volumes of solids of revolution, as well as computing arc length and surface areas. The study attempted to understand how mathematical fluency, defined as efficiency, accuracy and flexibility, and detected through the four parameters of foreign language learning (speaking, reading comprehension, writing, listening comprehension), is related to student success. This (student success) was tested on the operational level, and on the level of conceptual understanding, which was measured through the parameters, and by means of questions which explored the actual geometry involved in the applications and the idea of accumulation. The interview questions are included in Chapter IV. The interviews were audiotaped.

There were four interview questions that composed the fundamental body of the interviews, the first two consisting of three parts a), b) and c). They have been presented in their complete version in Chapter IV. The following presentation of the results of the interviews will be realized in three parts, the first two corresponding to the first two interview questions, and the third corresponding to the last two questions. The graphs of the 8 regions referred to in the first interview questions are in Appendix D and will be denoted by 1a, 1b, 1c, 2a, 2a', 2b, 2c and 2c'.

The Set Up

The students were asked to set up the integral that would permit them to find the volume of the solid in the first question, part a, which consisted of the following information about boundaries and axis of revolution $y = x, y = 4x - x^2$; about x-axis. The question was written on the blackboard, and they were told to read it, and begin to work as they would normally do, only that they should verbalize what they were doing. Nobody had any problem with interpreting the question, which means that the reading comprehension, after having done so many problems with the same language in their homework assignments, was good. However, the actual use of the terminology such as "bounds", "boundaries", "region", in their spoken mathematics differed. Whereas Tom and Paul both used the terms correctly and with ease, Susan had another attitude towards the terms:

> **Mariana:** What is the region?
> **Susan:** The area between the function and a line.
> **Mariana:** And what does bounded mean?
> **Susan:** When it starts going "the solid generated by revolving the region bounded…" all those big words, blah, blah, I just see the numbers and I try to solve what they want.

It is interesting to mention that Susan really did think that the region had to be between a function and a line and, as will be seen in the following problem 1b, this caused hesitation and confusion when the region did not have that characteristic. It is also interesting to point out the mathematical metaphor "I just see the *numbers*". It seems that she is referring to the whole algebraic symbolism, including the variables, as "numbers", and these "numbers" as opposed to "big words".

All the interviewees proceeded to try and graph the region, although their techniques for doing so differed, and so did their success. Susan drew the region without a calculator, but then used the graphing feature of the calculator to find the intersection points of the two functions, to be able to set the limits of integration. Laura immediately set out to find the intersection points before graphing. First she tried to use the table feature of her calculator and failed; then she tried to do the algebra by setting the two functions equal to each other, and could not do the algebra. Paul and Tom immediately graphed the two functions on their calculators (Paul is the only one who uses a TI-89, the others use a TI-83). However, whereas Paul also proceeded to find the intersection points, Tom did not do so, which turned into an issue when he needed to specify the limits of integration. The written work of this phase also varies in terms of accuracy and efficiency, as can be seen in the examples included in Appendix E. Whereas Laura spent two pages of trial and error, often proceeding because of my prodding, explanations, or outright showing her what to do, Paul and Susan sketched the region with its explicit bounds at the first attempt; Tom did the same, but without explicit bounds.

Through the students' speaking, the next steps, which consisted of deciding which method to use, and how to set up the integral, were portrayed. The extremes were Paul and Laura. Paul, through a monologue, showed his fluency:

> **Paul:** I could do two different methods of solving it, the simpler one would be the washer method. I'm going to set it up...the integral from zero to three, π, big radius which is $4x-x^2$, the little radius which is x, then squared, dx.

On the other hand, after going through the algebra to find the limits, and finding the intersection points with my help, the following dialogue with Laura took place:

> **Laura:** Zero and three. I'm very bad at drawing graphs.
> **Mariana:** The intersection points are zero and three if you are using which method to generate the volume?
> **Laura:** If you use any method, right?
> .....
> **Laura:** I need to revolve this about the x-axis. I'm using the disk method. The shell method is the only one that goes perpendicular to the axis about which it's revolving...
> **Mariana:** Parallel.
> **Laura:** So this is going perpendicular to the x-axis (she draws parallel). dx is that, this is the radius, the height...
> **Mariana:** What is the height in the disk method?
> **Laura:** It follows the curve of the graph.
> **Mariana**: That's the radius.
> **Laura:** The height, well this is the radius, right? (She shows the interval). The height would be x.
> **Mariana:** No, the motivation is the horizontal cylinder.
> **Laura:** Then what is the height of the cylinder, $4x-x^2$?
> **Mariana:** (*Drawing cylinders*)
> **Laura:** I get confused with the three dimensional, just draw the parallel and perpendicular rectangles

There are several aspects, related to fluency, that are portrayed through Laura's speaking. On the one hand, having to see the cylinders horizontally causes the basic notions of height and radius to become fuzzy and unclear. On the other hand, she has been trying to memorize the disk and shell methods according to the formulas (especially the "π" or "2π" attached to one or the other) and the rectangles being parallel or perpendicular, and far from clearing things up, this attempt at rote memorization has caused her to say things that hint at her lack of fluency in basic geometric concepts such as height, radius and the difference between parallel and perpendicular. It comes to mind to ask the question if, at the time she was learning these basic geometric concepts, she had proceeded in a similar way, in the sense of memorizing the formulas without understanding the geometry. In the case of the disk and shell methods, she could not visually relate the perpendicular rectangles to the slender cylinders (disks), and the parallel rectangles to the three dimensional shells. This could belie the lack of sufficiently solid schemas, which would have permitted her to formulate the mental models of "disks" and "shells". Her stage, at this point in the learning process, is that of "integrator". She does realize that the concept is new, but cannot relate it to familiar, well known concepts. These interpretations are reinforced by the following dialogue:

Laura: And the washer method is different?
Mariana: No.
Laura: Don't we need an extra h? (*she means "height"*)
Mariana: We need to take into account the two radii, but there is only one height.
Laura: I wish that three dimensional thing worked for me. Can we use a different color? Can you draw the 3 dimensional with a different color?
Mariana: (*Explaining and drawing the solution....*)
Laura: If I could make copies of this. I have a huge problem thinking conceptually, visualizing. Give me something to integrate and I'm fine.
Mariana: An algorithm.
Laura: Give it to me and I do it, as long as it's not trig!

The next problem was 1b which, in several ways, was more complex than 1a, although its "complexity" consisted in an expansive generalization, not a reconstructive generalization (Tall & Harel 1991, see p. 24 of the present study). In this case, the instruction remained the same, and the actual information about boundaries and axis of revolution was:

$y = x^2, x = y^2$; about $\mathrm{x} = -1$. The reconstructive generalization was related to the need to express the two functions in terms of the same variable, whether it was to be x or y.

There were several issues with regard to this problem, and the interviewees showed patterns in relation to these issues. First of all was the problem of expressing the function in terms x or y, which was related to the concept of inverse function, as will be described later on; the second was the axis of revolution. In two instances the students spoke $x = -1$ as they drew $y = -1$.

> **Paul:** (*Reads*) The two functions, $y = x^2, x = y^2$; about $\mathrm{x} = -1$. First, I would probably change the $x = y^2$ in terms of x, I believe that's what you call it (laughs), which would be $\sqrt{x}$, so $y = \sqrt{x}$ and $y = x^2$ about $x = -1$. Again I'd draw the graph of the functions on my paper, and where I'm going to revolve it around, which I'm doing right now. I'm labeling (*proceeds to draw* $y = -1$, *horizontal instead of vertical*).
> **Mariana:** The line is $x = -1$.
> **Paul:** Thank you, it's a "-1", but wrong one (*obviously embarrassed*). Now that I had that pointed out to me, it might be a little bit different way of solving the problem, um, so I'm going to figure out what they look like on my calculator.

Paul decided to leave the functions in terms of x,

> **Paul**: I can either do washer, but I'd have to change the functions, or it looks like I can do the shell method, and I'm going to set up for shell method, because I don't want to change the functions. The integral is

2π times the integral from 0 to 1, the radius which is x times the height which is square root of x minus x^2. (*the radius is really* $x+1$).
Mariana: If you had done the washer method, you would have had to take into account the -1, the fact that you're adding a distance of one. How about the next one...
Paul: No, this doesn't seem right to me, we're going around, well you're right, something is missing, the radius, we're going around $x=-1$...So the radius would be $x+1$.
Mariana: That was fantastic, because you caught yourself. When did you become aware of your mistake?
Paul: When you talked about the washer method, I said "wait a minute", it's not as easy as I thought.

The parameter that is key to explaining Paul's flexibility, one of the components of mathematical fluency, is listening comprehension. Paul was able to listen and understand my explanation of how the washer method "would have been" if he had chosen to use it. This triggered a doubt in his own mind about what he had actually done and his mental model of radius conflicted with what he had written down. Paul is at the synthesis stage as far as the operating and comprehension of the generation of volumes of solids of revolution goes. The concept possesses its unique identity, and is used for strategy development; his mental structure, when working with the notation, is that of a procept, as he can move back and forth between the integral as an instruction to realize the integration process, and the integral as a representation of volume, in this case. The integral, once set-up, is an instruction on how to operate, but at the same time it is the volume of a solid generated by revolution about a line. At the same time he committed an error which has been recurrent in my observation of calculus students, which is confusing the line $x=$ with the line $y=$. I would say that this is a cognitive obstacle with a didactical origin, as for several years students are taught that the

x-axis is the horizontal axis, and the y-axis is the vertical axis, and the association remains embedded in the students mind.

The following is a rather long transcription of Susan's reasoning process when confronted with 1b.

Mariana: What about this, (*writing*: $y = x^2, x = y^2$; about $x = -1$).
Susan: $y = x^2$, I just plug this in. Then $x = y^2$, it's just the same thing, right?
Mariana: Well, in terms of which variable are you going to work?
Susan: I'll do it in terms of x. It's probably something like that (realizes a perfect drawing). And then about the negative one (*she draws* $y = -1$, *instead of* $x = -1$).
Mariana: About $x = -1$.
Susan: Oh, I remember now, so this is at zero...then this would be the distance from zero to negative one is one, and then from one to two is one (*she keeps drawing in terms of the horizontal axis*).
Mariana: About $x = -1$.
Susan: O.k. (*she doesn't pay attention and still draws horizontally*).
Mariana: $x = -1$.
Susan: Then it's going to be rotating here (*she draws the vertical line*) like a mirror.
Mariana: Yes! How would you set that up?
Susan: It's a washer, right?
Mariana: It could be. If you want to use the washer method, do you set it up in terms of x or in terms of y?
Susan: I did it in terms of x because actually, wait a minute, it's rotating about ...this is like y.Oh o.k., you do it as if it was rotating around the y-axis. So it's going to be the function minus one.
Mariana: So, are you going to use the washer method?
Susan: I did, but this is not a straight line, like we used to have.
Mariana: That's o.k., you can take a function as your big radius. In the other case your function happened to be the line $y = x$, but it was still a function. If you use the washer method, and you are going to revolve about a line that is parallel to the y-axis, how do you set up the integral, in terms of x or in terms of y?
Susan: In terms of y.
Mariana: And your big radius?
Susan: In terms of y, so it's x square (*as she writes it*).
Mariana: In terms of y.
Susan: Square root of x? (*She writes it down*).

Mariana: In terms of y.
Susan: If I put y^2?
Mariana: If you have $x = y^2$, then the other radius function has to also be in terms of y.
Susan: Square root of y and y square.
Mariana: And which one is on the top, and which on the bottom?
Susan: I don't know how to graph square root of y. (*She refers to the calculator*).
Mariana: Oh, that doesn't matter, it's analogous to square root of x. You can only use x on the calculator.
Susan: I put this one as the big radius because it's further away.
Mariana: In terms of y, which one is on the top?
Susan: But the one that's furthest away is on the bottom.
Mariana: No, furthest away in terms of the line $x = -1$.
Susan: Oh, I see. In terms of x it's x square, but in terms of y it's square root of y.

The first aspect that will be pointed out is the same error as was shown with Paul, that is, the line $x = -1$ drawn as the line $y = -1$. In the case of Susan, even when it was pointed out, it took a while for her to understand that an error was being indicated. In terms of the parameters, this is a writing mistake, which belies a lack of fluency, in particular in the component of accuracy. However, this mistake, as pointed out above, is frequent and points to a cognitive obstacle which could and should be studied in itself. When she finally did realize that the line $x = -1$ was parallel to the y-axis, her reaction was very different from Paul's. She did not seem to think it had been an important mistake, but showed interest in the new perspective, seeing this rotation "as a mirror". Through this spoken extra-mathematical metaphor, two things were detected. First, the mirror metaphor is used for the revolution about a vertical line but not about the horizontal line, and second, the actual revolution which generates volume, is seen as a reflection in the plane.

Susan expressed the mistaken idea that to use the washer method, the big radius had to be a line. Many of the practice problems did have this characteristic, but by no means all of them. In class, the students had worked on a specific problem in which both radii were curves different from the straight line. Through Susan's spoken and written mathematics, certain mental models created in the process that reflected very peculiar schemata, often full of errors, were detected. At the same time, Susan also was very flexible once the error was pointed out to her, and sometimes on the second "try", more often on a later "try", she would get to the correct procedure, or express the precise concept. The whole process of interviews, plus her class participation, in which she constantly asked questions, revealed her to be in the analytical stage, according to the four stage classification.

Another problem detected through the writing and reading comprehension parameters, and stemming from mistaken schemas that translated to badly constructed mental models, had to do with the whole issue of expressing the functions "in terms of x" or "in terms of y". As was mentioned previously, the inverse function, which comes from the algebra and precalculus curriculum, is at play when making the decision of how to express the functions that will define the region. The construct of procepts also helps to understand what is occurring, and how this relates to problems with mathematical fluency, in particular accuracy and flexibility. In the particular interview question, the functions are presented as $y=x^2, x=y^2$, and it is not clear for all students, even at this stage of calculus II (when it is assumed that their understanding of functions is complete) that $y=f(x)$ and $x=g(y)$. This was corroborated in the classroom, when I was teaching. The symbols $y=f(x)$, as "synonyms" seemed to be natural, after so many years of routinely manipulating them as such (going back to the straight line introduced as $y=mx+b$, and then treated as a function in algebra), but $x=g(y)$ was not intelligible for at least half the class (this was determined when I put the expression on the blackboard as we were working out a similar problem, and asked the students who did not understand the relation to raise their hand. I'm not sure that those that did not raise their hand really did understand). The interviewees also did not respond, when the idea of inverse function was suggested. Instead of being seen as a tool which could clarify any confusion (taking into account that the idea of inverse function is introduced in algebra) the use of the concept of inverse function was seen as adding to the confusion. This was openly expressed in the lecture session, when I asked for the students' opinions to that respect. A surprising lack of metaphors in relation to the idea of inverse was detected, in spite of how this concept lends itself to extra-mathematical metaphors such as "returning", "undoing", and "opposite". Finally, the calculator doesn't help to clarify these issues, as the variable x on the calculator was seen by Susan as *the* x, and not as *any* variable, depending on the specific set-up and, consequently, she expressed that she didn't know how to graph a function in terms of y. The lack of understanding of inverse functions also caused problems in the geometric aspect of setting up the integral.

This was detected through the listening and reading parameters, as can be seen in the following example. The big and small radii were expressed as the "top" and "bottom" functions, but seen and drawn in terms of the horizontal axis instead of the vertical axis (in other words, the two students who did this expressed the function algebraically in terms of y, but drew it in terms of x). When it was pointed out by means of my writing (drawing) that the distance was in horizontal terms of "how far" from the line $x=-1$, the student could proceed, and finally set up the integral. In this case, their reading comprehension (of my writing) was what made the problem clear, and the listening comprehension was very poor.

The next question was 1c, which also asked for the set-up of the integral which would represent the volume of the solid of obtained by revolving the region $y=\sin x, y=0, x=2\pi, x=3\pi$ about the y-axis. In this case three of the students interviewed revealed a lack of local fluency in trigonometry. Tom had no problem in understanding the bounds of the region, and no need of my help or prodding during the whole problem. Once Paul's trigonometry issues were resolved with my intervention, both Paul and Tom showed a quick understanding of the set-up itself, revealing mathematical fluency, especially in terms of the efficiency component, when distinguishing the essential differences, both geometric and algebraic, between the disk and shell methods. Tom, in this case, showed immediate reading comprehension, as well as mathematical fluency in all its components when speaking and writing.

> **Tom:** Let's see (*draws sine function, finds bounds*), it goes here, so we're looking at $2\pi, 3\pi$, this guy here, around the y-axis, so this dude here, so the shell method is the easiest way to do it, from 2π to 3π, radius is...I think that's it.

Mariana: Great, so you decided on the shell method, which was the only way. If you turn it around (*the paper*) you can't integrate because...
Tom: It's not a function; it doesn't pass the vertical line test.

Paul also was efficient in dealing with this problem:

Paul: My trig is really bad, I haven't taken it...(*explanation*).
Paul: All right, now that I have that sorted out, I draw my graph, and I have the little hill kind of thing going around in a circle about the y-axis, let's see, so I could set it up doing the shell method, so from 2π to 3π (laughs), that was a little confusing, my function is $\sin x$, so I have the radius x, because its about the y-axis and we don't have to mess with it this time, times $\sin x dx$.

On the other hand, Susan's lack of local fluency in trigonometry and algebra (as determined by her constant mistakes even when in the pure algebraic or trigonometric context as opposed to the global one of the interview questions) contributed to her difficulty with the new material of applications of the integral, in which her stage oscillates between integrator and analytical, depending on the type of problem. When it involves trigonometry, even concepts that in other problems seemed to have been clear, are ill defined and fuzzy.

Susan: Do you want me to say, this method I'm using is the shell method, or the disk method, things like that...
Mariana: It will be obvious when you set it up, you don't have to say it beforehand. And how are you going to get that volume?
Susan: R squared the radius. Do you want me to go from 2π to 3π?
Mariana: It depends on what method you use what your limits will be. In terms of x yes those are your limits, in terms of y they are not.
Susan: In terms of y I use x, remember?
Mariana: In terms of y you are saying that $x=$, it is a function in terms of y.
Susan: If I square the radius, I use the disk method, and it's the really washer method.
Mariana: And how do you get this function ($\sin x$) in terms of y.
Susan: You just switch the x and the y.
......
Mariana: From here to here is a function, with the trigonometric functions

you need to be careful, you need to specify the domain...
Susan: What's the domain?

......

Susan: So why did you make me go around the y-axis if we can't solve it.
Mariana: Of course we can, we don't have to use the washer method.
Susan: The shell method, but can I use it from 2π to 3π? (*it seems that everything is blurry since the week before, as in class we have been doing different material such as arc length, surface area and work applications*).
Mariana: Yes, and what is the radius?
Susan: π, because I'm going from 2π to 3π. (*She has forgotten that the radius is variable*).

Susan's lack of mathematical fluency is portrayed in this example through her speaking, writing, listening and reading comprehension (or lack of it). In particular, accuracy and efficiency have been lost from one week to the next. This can only be explained in terms of very weak schemata. Once another element is introduced, in this case trigonometry, in which her local fluency is weak, her mental models and strategies are full of contradictions, and are meaningless geometrically. She also has tried to memorize rules for the use of the disk and shell methods:

Susan: Mariana, I have a question. In the disk method, the x is what we're revolving about, if it's x its dx, if it's y, it's dy in the disk and washer method. Now the shell method, if it is revolving about the x it is dy, and about y, dx.

Memorization is often useful in the learning process of any subject, including mathematics. However, from my perspective, in Laura's case the memorization was seen as a way of avoiding any kind of geometric analysis or thinking, and in Susan's case, although to a lesser degree, it was also seen as a way of avoiding the actual geometric reasoning when setting up the integral. When this happens, it seems that the mathematical content which is supposedly mastered from previous courses, and should help to solidify the new material both conceptually and operationally, is not only not used in this way, but is avoided precisely because it has not been mastered, or even understood. It is not reasonable to expect fluency in these cases, even when so much effort is put into memorizing rule after rule because, as was quoted in Chapter II, "...students in many educational programs fail to achieve fluency. Instead, they progress by building one non-fluent skill on top of another until the whole skill set becomes 'top heavy' and falls apart." (Bateman, Binder & Haughton, 2003, p.2).

Working Backwards

This section corresponds to the second problem parts a, b and c. In these problems the students were presented with volumes and areas expressed in terms of integrals, and were asked to make sketches (very rough in the case of the volumes) of what these volumes and areas looked like. Part a) consisted of

$$V = \pi \int_0^4 (\sqrt{x})^2\, dx \qquad A = \pi \int_0^4 (\sqrt{x})^2\, dx$$

The problem was given to Tom at first without any explanation. He began to sketch, and then I spoke:

> **Mariana:** Now we're going to work backwards...As I wrote, I just want you to sketch...
> **Tom:** That looks like fun. From zero to four (*sketches the volume perfectly*).
> **Mariana:** Now if I was to say that I'm looking for area, not volume with this same integral, it will give us the same number.
> **Tom:** That's interesting, the same number, the bounds are the same (*draws the line, but as* $f(x)=x$, *not as* $f(x)=\pi x$).
> **Mariana:** (*points out error*). The integral is just the number, it depends on the units that are attached, how it's generated, it's meaning...
> **Tom:** I never thought about it like that. For every area there's a corresponding curve. That's kind of cool. The volume of one function, when you look for the area, it turns into another curve; it has its own sister function. I find that interesting.

When the problem was given to Paul, an interesting process occurred. He proceeded without really paying much attention to the actual set-up of the integral, and once he sketched the region, he generated the volume with the figure that he had in mind. This could be classified as a process in which the metaphor (his preconceived figure), in this case geometric, was counterproductive.

> **Paul:** I'm gong to draw my x, so that I have a little frame of reference. The function would be square root of x, like this, from 0 to 4, so you kind of have this shape, that would mirror it (*he revolves about the y-axis instead of the x-axis*), it looks like the donation things they have, like a tornado that goes in.
> **Mariana:** You're rotating about the y-axis, that means that if you set-up in terms of x, you would be using which method?
> **Paul:** I would be using the shell method.
> **Mariana:** And the way it's set up?
> **Paul:** That would be disk method, oh, I know what I did, I kind of did it the way I liked to do it, but it would be going around the x-axis, I kind of have a solid parabola. So it would be disk method. I kind of ignored everything, I just did the graph, oh, "here's the graph" without thinking. I don't know what I was thinking.
> **Mariana:** It's hard to work backwards. What would the area look like?
> **Paul:** πx is a line.
> **Mariana:** With a slope of...?
> **Paul:** π.

Susan, on the other hand, struggled much more to understand what was expected.

> **Mariana:** I want you to sketch what it looks like, so we're going backwards.
> **Susan:** Can't I just plug in?
> **Mariana:** I have a volume that's equal to π from 0 to 4, times the square root of x, squared. I want to see if you can go backwards and from this integral give me a sketch of the volume.
> **Susan:** Can I eliminate the square root of x, squared, and just put x, it would be easier.
> **Mariana:** If you do that, you loose the formula for the volume.

For Susan it was very difficult to understand the difference between the original functions that served as bounds for the region, and the new function that generates the solid, and squares the original function in its role as the radius. When she was asked to find the area, and told that in that case it was all right to express the function as x instead of $(\sqrt{x})^2$, she had trouble understanding that π as a constant was just the slope of the line, and did not in itself signify volume.

Susan: Why is the π there, if it's just an area.
Mariana: I have the function $f(x) = \pi x$, π is just the slope of the line.

Whereas Tom and Paul were able to move freely between the integration symbol as a process (the evaluation) and as a concept (representing volume, area, arc length), Susan was not at the procept stage, and wanted to treat the actual volume through the process of integration, which she calls "plugging in". Even though it was explained that by integrating she would get a number that represented volume, but would not be able to identify the shape of the solid, the following parts of this question (2b and 2c), which were presented to her during the second interview, show that there had been no listening comprehension.

The next problem, 2b, just asked the students to sketch the volume. As presented, the integral $V = \pi \int_0^2 [16 - (y^2)^2] dy$, required a little more than the previous integral, as 16 was presented instead of 4^2.

In this case Tom lost sight of the concept itself, and began to work procedurally, with the following result:

Tom: Oh boy, all right this is tricky, $16 - y^4$ so (*he tries to express it in terms of* x*, as* $y = \sqrt[4]{16 - x}$ *and graph it on the calculator*), it doesn't want to show up.
Mariana: The function is set up so that we can get volume, but it's not the original function. What would the original function be?
Tom: Well, it looks like you're revolving about the y-axis because it's in terms of y.
Mariana: It could be the shell method in terms of y, but about the x-axis.

Tom: That's true, it's tricky. But it's got a π in front, not a 2π, so it has to be the disk method. Also, it could be that 16 is the big radius, and y^2 squared is, the small radius, or 4 is the big radius. That makes it less complicated. Then y^2 is the other function and $y=4$ is this guy (*it should be* $x=4$), that's cool. I didn't see it at first.

Tom is in the synthesis stage in terms of his relationship with the concept, as he is able to develop strategies and even make new allegories. However, he did show hesitation, and committed an error that could point to a still unstable procept in terms of moving freely between the integral as a process and as a concept. It was his reading comprehension and writing which portrayed his lack of mathematical fluency, in terms of accuracy, in the beginning. However, his speaking showed a large degree of flexibility, another component of mathematical fluency, as well as his listening comprehension and, with some prompting, he was able to change his strategy. On the other hand, it is interesting to note that in this case he also confused the lines $y=4$ and $x=4$. It seems that when certain confusion presents itself due to a problem which is less standard than usual, this confusion extends to local fluencies that, in other problems, had been used impeccably.

In Paul's case the following is seen:

Mariana: What about this one, if I was to ask you to sketch it.
Paul: Backwards, it's like walking backwards. It's the disk method again, yeah, but it's going around the y-axis, it's going... this is hard, you know, normally it's 16 minus, so I'm going around a line, not one of the axes, right, yeah, so it would be going around... (*silence*).
Mariana: You mean, there's a line involved, but you are going around one of the axes.
Paul: Oh, okay, I see. I get it, it's the washer method and the line $y=4$.
Mariana: $y=4$?
Paul: Oh, well I have $x=4$. It's not easy going backwards and doing the y-axis. I drew it right, I just didn't think of it right. So $x=4$ and y^2 is almost like x^2, but 90 degrees.

Mariana: Actually it's the square root of x
Paul: Yes, the square root of x. And so I'd have, like that, and it is going Around the y-axis, so I'm trying to think where the limits are going, so from $y=0$ to $y=2$.

Paul also is in the synthesis stage as far as this concept goes and can develop strategies but, as in the case of Tom, showed hesitation when confronted with an unorthodox presentation of what he would have recognized right away as the washer method used for revolving the region about the y-axis, had the 16 been presented as 4^2 (he admitted that if he had seen it that way he would have recognized it as the radius squared). Once this unusual element of presentation was introduced, even aspects which had been dealt with successfully in previous problems (expressing the function in terms of y, both algebraically and geometrically) were not clear. In the two cases, it was the reading comprehension parameter that showed the lack of flexibility, one of the components of mathematical fluency. They did not interpret 16 as 4^2. At the same time, both students showed flexibility in terms of changing strategies, as could be detected by their speaking, listening subsequent and writing. Susan's case was very different. Her listening comprehension seems to be very deficient, as the interview transcription in 2b almost seems to be a copy of the one presented for 2a.

Mariana: This we also did last week. I just want you to sketch what the following integral should look like.
Susan: Can I get the antiderivative out of that first, and then plug it in?
Mariana: You'll get a number, but I'm asking for a sketch.
Susan: Can I change y^2 to x^2?
Mariana: Oh, in the calculator you have to, it doesn't accept y.
Susan: I just got two lines. (*She did the same as Tom, and graphed the set-up for volume, not the functions that define the region*).

Mariana: This integral is set up to get a volume... It's revolving around which axis?
Susan: the x.
Mariana: Are you sure?
Susan: The y? (*she checks the chart she made the previous week*).
....
Mariana: What are the bounds of your region?
Susan: From zero to two.
Mariana: From zero to two are the limits, the domain. I want your region.
....
Susan: How do you know it's the washer and not a disk?
Mariana: You suggested it! Does it have a hole?
Susan: So that 16 came from 4, The big radius is 4 an the little radius is y^2.

The manifest problems in Susan's listening and reading comprehension, measure as well as contribute to a tremendous lack of efficiency in her work. The lack of agility in moving back and forth between the process and the concept (procept) in the integration process was as marked as the week before, and she still wanted to "plug in to the antiderivative" when asked to make a sketch. Once again it is clear that her schemata with respect to the function concept and the inverse function are weak. At the same time, she is very perseverant, does not tire, and will ask questions until arriving at the answer, which can be seen in the last three lines. This behavior is very different from Laura's, whose frustration was so marked, especially in relation to the geometric component, that I decided not to insist on 2a, 2b and 2c during her interviews.

In part 2c the students were asked to sketch: $V=2\pi\int_0^1 x^2dx$ and $A=2\pi\int_0^1 x^2dx$.

All students, at this stage, identified the shell method (by the 2π), and realized that the boundaries were $y=x$, $x=1$ and $y=0$. Susan still struggled with the area concept:

Susan: Why is 2π there if it's not a volume?

Accumulation and The Fundamental Theorem of Calculus

Question 3 was related to arc length, and the derivation of the formula to find it. The question was presented as: How do you determine arc length by using an integral? Can you explain, and/or draw a picture, showing how the formula $s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$ was developed? In three out of the four interviews there was a common conceptual mistake. This consisted of seeing the little straight line segments, which are "summed" under the integral sign to calculate the length of the arc of a particular curve, as "derivatives". Through this common mistake, detected by the speaking parameter (because their pictures, which would be considered as a writing parameter, seemed to correctly portray the concept) it was seen that students often refer to the "tangent line" and the derivative as if they were the same.

> **Mariana:** Why should we use the integral to determine arc length? We have all these tiny straight lines (*referring to the sketch*). What does the integral do for us?
> **Susan:** It tells us the antiderivative came from this derivative, each little distance is the derivative, that's what I think, each little line is the derivative.

> **Mariana:** We also used the integral sign for arc length. Why? What were we summing there?
> **Paul:** We were taking the slopes at different points of the curve.

> **Mariana:** Why do we integrate to find arc length?
> **Laura:** Because we are summing up infinite lengths of derivatives.

Paul did correct himself as he went through the process:

> **Mariana:** Where does the square root come from?
> **Paul:** From the distance formula, oh, we're summing up little straight line distances, not slopes.

The fact that the integral was being used to realize the sum of the straight line segments seemed to be clear, perhaps because I emphasized this when I introduced the material in the class. However, once again it wasn't until I talked about attaching units that the fact we were using the integral to obtain results in three different dimensions became explicit. It was through the listening comprehension, together with the speaking parameter, by which different degrees of mathematical fluency in relation to this more abstract aspect became evident. Even when students are accurate and efficient, their performance can be mechanical and not reflect conceptual understanding. When all the parameters are used to measure fluency, it is easier to uncover deeper problems of conceptual understanding, which will turn into obstacles as the student advances in mathematics and its applications.

The next question was of a more descriptive nature, and the students were asked why they thought the integral was used to calculate areas, volumes and arc lengths. The answers varied:

> **Mariana:** Did you ever think about why we're using the integral for so many different purposes, to find area which is two dimensional, volumes which are three dimensional, arc lengths which are one dimensional, surface areas, work?
> **Susan:** The integral? You always ask "how did this come about?". I usually think "someone told me to do it this way" and I do it. You told us that we use the integral, I don't know, it's the derivative that gets it closer to the function by lines, the antiderivative, isn't that the opposite?

On the other hand, Tom answered in a very precise, if not totally accurate manner:

> **Mariana:** Did you ever think about why we're using the integral for so many different purposes, to find area which is two dimensional, volumes which are three dimensional, arc lengths which are one dimensional, surface areas, work?
> **Tom:** Because in every case we are summing up the little divisions, you know, the dx.

An extreme response was:

> **Susan:** I don't care if it's an area or a volume. I just want to integrate like I did with derivatives, and get an answer. I hate geometry.

Finally, the fourth question was developed to relate the previous concepts, introduced at the beginning of the calculus II course, to the Fundamental Theorem of Calculus that, supposedly, students have studied at the end of the first calculus course, and have mastered sufficiently to be able to find coherence in the sequence. The question itself was presented as:

The derivative can represent a rate of change. If a function represents a rate of change, it is the derivative of another function. Let f' represent the rate of change function.

What does $g(x) = \int_a^x f'(t)dt$ represent?

What does $\int_a^b f'(x)dx$ represent?

What is the difference between the two integrals?

My expectation was that, independently of whether they remembered the Fundamental Theorem of Calculus or not, they would identify the first integral as a function, and the second as a number.

In all four cases, and then corroborated by asking other students in the class, no-one could reproduce The Fundamental Theorem of Calculus, not even the first version, until I asked what they would do with $\int_a^b f'(x)dx$. Then, all answered in more or less similar terms that they would just "plug in" a and b to f. All students asked said, without exception, that they had never seen anything like $g(x)=\int_a^x f'(t)dt$ (in terms of having a variable as a limit), until I showed it to them in the book (Larson, 2003, p. 316). However, independently of the memory aspect, which must be studied in a broader context to see if students retain the general idea of the second version of the Fundamental Theorem of Calculus (as it seems they do with the first version, because they constantly use it), the interpretation of the integral $g(x)=\int_a^x f'(t)dt$ is very related to reading comprehension, as once the variable was "plugged in" to the $f(\ )$, they realized it was a function.

> **Mariana:** What's the difference between the two integrals?
> **Paul:** Constants and variables.
> **Mariana:** Right, whatever the integral represents as area, volume, arc length, when it's set up as $\int_a^b f'(x)dx$, it represents a...?
> **Paul:** Number.
> **Mariana:** And the other one?
> **Paul:** It's an integral, it can change, x could be anything. It's like a function.

> **Susan:** From a to x, I've never seen that. It's x to a instead of b to a. Is it going to be like a change, because the x is changing?
> **Mariana:** Right! That's very important.
> **Susan:** If it's from a to x, it's from any number between a and b to x.
> **Mariana:** Exactly from a to x, as you said at first.
> **Susan:** That means that x could be anything, as long as it's between a and b. But you don't have the b, so how do you know that the x you choose is between a and b?

Once again, Susan's questions place her in the analytic stage in the context of the concepts that this problem deals with, and in this case her reading comprehension permitted her to understand the deep underlying difference between the two integrals, in terms of one being a constant and the other a function. However, she did not see this as efficiently as Paul and Tom did, as can be seen from the reading comprehension and speaking parameters of their interviews.

Laura was suspicious of the question, as she had previously expressed:

> **Mariana:** Do you remember how you were first introduced to the integral?
> **Laura:** You know, I don't care what the integral means. I don't care if it's an area or a volume, I just want to integrate. That's why I liked calculus 1, it was just doing derivatives and stuff.
> **Mariana:** What about the applications, optimization for example?
> **Laura:** The word problems? I hate them. I hate applications. I just want to do derivatives, integrals; I don't care what they mean.

However, once she realized that by "plugging in" the limits, she could distinguish between a "$f(x)-f(a)$" as a function, and the constant "$f(b)-f(a)$", she expressed:

> **Laura:** That's cool. And the "$-f(a)$" is just a particular C.

Her reading comprehension and writing showed a depth of understanding as she was the only one who related the whole expression to the concept of antiderivative, once the difference was pointed out.

The interview with Paul lent itself to a question on a more metacognitive level, as can be seen in the following transcription:

> **Mariana:** When you saw it for the first time, and were told that the Fundamental Theorem of Calculus show there is a relationship between derivative and integral, because they talk about the antiderivative, did it

seem meaningful?
Paul: Um, a little bit. I don't really think it was, we were, like, told some of the relationships but, what it meant, I didn't know. It confuses the heck out of people, but it's a good thing to talk about every now and then. Not just once or twice and drill it in someone's head, but maybe go back over it again in calculus 1, 2 and 3. That would make people understand it a little more and understand what we're doing.

Summary

When students begin the standard calculus II course, the professor will expect that they are equipped with previous knowledge from calculus I that is used, implicitly and explicitly, when they are confronted with the section on applications, usually presented at the beginning of the course. It is also expected that they have a certain degree of geometric intuition, as well as fairly sophisticated algebra and trigonometry skills, and that they are capable of making connections and extracting meaning out of the tasks they perform (for example, realizing that they are using the integral as a means of accumulating infinitesimal volumes, areas or lengths). When talking about success with applications, and underlying notions such as accumulation, in relation to the process of integration and the concept of integral, the study showed that, by using the four parameters of foreign language learning (speaking, writing, listening comprehension and reading comprehension) to measure mathematical fluency as defined in this work, it is possible to detect and explain this success or failure in the processes, and relate it to understanding at various levels. Through a classification of students' expressions when speaking, writing, listening or reading, it was possible to get a firmer grasp on their reasoning processes, independently of whether these processes were coherent, and led to correct performance and conceptual understanding, or whether they were mistaken. In particular cases, which have been mentioned in this chapter or will be in the following, once certain tendencies were suspected to exist, it was possible to test in the context of the classroom, to see how the other students performed or showed their understanding when confronted with similar situations. This was usually done through class exercises, actual questions to specific students, and analysis of homework problems and the exam. In the chapter on conclusions and recommendations, it will be suggested how curriculum could be adapted to the scheme of the four parameters, and how this might enhance, on the one hand, the learning process, and, on the other, the feedback which the teacher or professor receives about the actual learning and understanding that is occurring.

The study showed that the students who possessed adequate procepts in terms of the integral symbol itself (as representing an instruction to integrate, as well as an actual volume, surface area, arc length, or whatever the context required) had a much higher degree of success when going through the actual process, whether this success was reflected in a precise numerical answer, or a correct description of the processes or concepts. Whereas Susan and Laura searched for rules and algorithms that they could memorize, and treated the integral itself as an instruction to proceed with these rules (and were frustrated when procedures were not always clear from the prototype, such as when the region was to be revolved about a line different from one of the axes), Tom and Paul seemed to have no problem distinguishing between the process of integrating and the actual applications. This was especially clear when students had to "work backwards" from an expression representing volume or area, and previously set up as an integral. They were asked to make a rough sketch of the original region and the volume of the solid generated by revolving it about a line, or of the area under the curve, with the same integral. The interviews, by means of using the four parameters to measure fluency, showed that it is unreasonable to expect reading comprehension if fluency at the local levels, especially the flexibility component, has not been achieved. This differs very little from what would be expected from reading comprehension in foreign language. If the student does not count with the "vocabulary" and the conceptual structure, he or she will not comprehend the new material. Susan, who just saw the integral (not the volume or area) had read the problem aloud for me, and to herself, and would have proceeded to just integrate ("plug in") if the speaking parameter, instigated by the actual interview situation, had not made me aware of her intention. Poor listening comprehension reflected the absence of mathematical fluency, and more specifically the lack of efficiency, given the fact that this exact same conduct was repeated when she was confronted with analogous questions the second and third time around.

When explaining the derivation of the formula for arc length, the sketches of all the students, considered as a writing parameter, showed an understanding of the concept of accumulation of infinitesimally small straight line distances; this was probably due to the emphasis that I had made on this aspect during the class presentations. Their performance reflected good listening comprehension or, in its defect, good reading comprehension of the notes they had made and studied. However, through the speaking parameter a common conceptual mistake was detected, which consisted of seeing the little straight line segments which are "summed" under the integral sign, to calculate the arc length of a curve, as "derivatives". In other words, instead of relating to the distance formula, their concept image reflected the sum of the tangents of all the little lines. This would not necessarily have been detected without the speaking parameter. This particular problem was then detected in approximately half the class when they were presented with the same question, and is hypothesized to be related to the existence of a cognitive obstacle stemming from the manner in which the right triangle is used to explain the slope of a straight line, as well as the distance formula and the trigonometric relationships.

A very interesting extra-mathematical metaphor was coined by one of the interviewees when dealing with the need to make a sketch of the solid (generated by an original region), and the area under the curve, of the same integral. This student referred to "sister" functions, that as abstract integrals produce the same number, but whose origins are as different as a curve and a region (area). In terms of mathematical fluency, this apparently playful metaphor reflected a great deal of flexibility in his thinking process, which coincided with his general performance and understanding in relation to the interview tasks. On the other hand, manifestations related to the constant π showed, in the case of other students, that the distinction between the integral when representing area under the curve and volume (without mentioning arc length, surface area, work) was very weak. The presence of the constant π was associated, in a mechanical way, with volume or, even more, with the implementation of the disk or shell methods. One of the interviewees did not want to talk about volume, just about parallel and perpendicular rectangles associated with π or to 2π. The reverse situation also presented itself in the classroom setting, when after having gone through the material on applications, and reviewing for the exam, a problem concerning the area between two curves (finding regions) was being worked on. One student who did not count amongst the interviewees, but with whom I worked with frequently, suggested multiplying by π, as if the professor had forgotten to do so. When it was pointed out to him that it was only the area between the curves that was to be calculated, not the volume of a solid generated by revolving the region about a line, he admitted that he had not really understood, up until then, what had been going on in the whole section on applications, even though his procedural performance at a given moment was adequate. This reinforces the importance of the use of the four parameters of foreign language learning to detect student understanding, as in this case, although the accuracy component of mathematical fluency was good, the flexibility component was very weak, and very possibly would have inhibited further progress.

In brief, this study has shown that the four parameters of foreign language learning are useful tools for detecting mathematical fluency, and that there is a pattern to the key problems that students encounter which can be described using the terminology of certain theoretical constructs developed by researchers in mathematics and collegiate mathematics, as well as cognitive sciences, such as *procept, schema and mental models, extra-mathematical and structural metaphors* and *cognitive obstacles*. It is also possible, using the parameters of mathematical fluency, to locate the key differences among the fluent students who experience success with applications (in terms of accurate answers and efficient procedures) and show understanding of the underlying notions (which implies flexibility in thinking, and problem formulation and solving) when confronted with the process of integration and the concept of integral, and those non-fluent students, who are not achieving success nor understanding.

CHAPTER VI

CONCLUSIONS

This work had, as its objective, the study of the relationship between students' performance and understanding with respect to the integral as a process and as a concept, and mathematical fluency as it is defined and presented in this context. The methodology relied heavily on the use of the four parameters of foreign language learning, that is, *speaking, writing, listening comprehension* and *reading comprehension* to exhibit mathematical fluency, whose components, as defined in this study, consist of accuracy, efficiency and flexibility. Efficiency was understood as having developed schemas related to the different mathematical areas, enabling the development of strategies that translate into less time and less effort when confronting problems in these areas. Accuracy was understood, in the numerical context, as arriving at the correct answer; in the symbolic sense as the correct usage, manipulation and interpretation of mathematical symbols, according to standard mathematical norms, and in the spatial sense as the ability to correctly perform anything from basic geometric operations (such as rotations, reflections) to more sophisticated spatial operations, and to distinguish, identify and situate objects in a spatial context, according to standard mathematical norms. Flexibility was understood as the ability to recognize when a strategy doesn't work and is not applicable, and to change the strategy at that time in the process. Flexibility requires conceptual understanding, and the concept of mathematical fluency, as defined in this study, cannot be divorced from conceptual understanding. The elements of the five central components of the theoretical framework: duality process-concept; schemas, mental models and strategies; four stage model of mathematical learning; metaphors; and cognitive obstacles, gave the language and concepts to analyze mathematical fluency (efficiency, accuracy and flexibility) as indicated by the parameters of foreign language learning.

The methodology employed in the study was a combination of action research and interviews, and the setting was a standard calculus II course, team taught, at a community college. The research questions, which guided the study and will set the tone for these conclusions were as follows:

Research Questions With Respect to Mathematical Fluency

1. What is the relation between students' comprehension of written mathematics (reading comprehension), and mathematical fluency?

2. What is the relation between the logical structure, method of attacking problems, sequence of steps and sketches in students' written mathematics, and mathematical fluency?

3. What is the relation between students' spoken mathematics and mathematical fluency?

4. What is the relation between students' comprehension of spoken mathematics (listening comprehension) and mathematical fluency?

Research Questions with respect to Performance:

Applications

1. What stage(s) does the students' reasoning reflect when dealing with applications?

2. What are the key metaphors used when having to deal with applications?

3. What role do cognitive obstacles play when students are asked to set up and use the integral for applications different from the area under the curve?

Fundamental Theorem of Calculus.

1. How do students deal with the process-concept duality when confronted with the symbols of this part of calculus?

2. What stage(s) does the students' reasoning reflect?

The detailed results themselves, present answers to these questions. In this chapter, the results will be synthesized and interpreted in the terms of the research questions, and implications based on these conclusions will be stated.

Measuring Mathematical Fluency

When analyzing the transcripts of the interviews, as well as the observation notes of the class, the four parameters were artificially separated, as they are when analyzing student performance in foreign language. Once this was done, the material was grouped according to the general rubrics covered by the interview questions, and the natural connections between the parameters and fluency were recovered. However, the initial separation was very important, as it permitted the observation of speaking, writing, reading and listening comprehension on an individual level.

The relation between students' comprehension of written mathematics and mathematical fluency is constantly tested in any academic situation. It is expected that the student, after a lecture session, will be able to read the relevant material in the text, even aspects or variations that were not specifically covered in the lecture, and follow the instructions that are given with the problems, that are usually assigned as homework. The way this reading comprehension is tested is by evaluation of the students written work, which is an example of why, at the end, it is necessary to make connections between the different parameters. As in the foreign language, reading comprehension is diminished or nonexistent when prerequisites such as vocabulary and definitions, and practice in making sense out of them, are absent. If there are no schemas, or the ones that exist are very weak or incoherent, the student will not be able to construct the mental models that give meaning to the material being read. If the student is at the allegoric or integration stage, then the relationships and connections to known concepts that are being taken for granted in the text, or in the instructions given in a problem set or an exam, are not at all clear. When the students were presented with the two integrals $g(x) = \int_a^x f'(t)dt$ and $\int_a^b f'(x)dx$, and asked to specify the difference between them, their reading comprehension in mathematics can be explained in terms of having developed, or not, procepts. Even if the students do not remember the Fundamental Theorem of Calculus, it is supposed in the curriculum that their years of experience with algebra and precalculus, as well as with reading the symbolic language of mathematics at the levels through which they have passed, is enough to indicate that one integral represents a function, and the other, a number. There were other instances as well where reading comprehension was poor, if the procept had not been developed. For example, when students were asked to "work backwards", in the sense of sketching the region and the volume generated by this region of a particular set up, say, $V = \pi \int_0^4 (\sqrt{x})^2 dx,$ if they just saw the process indicated by the integral sign, and not the actual volume, they wanted to evaluate the integral, ignoring completely the instruction to sketch. This is a clear case of the particularity of reading comprehension in mathematics, in which it is not enough to have the reading comprehension of the words in the given language where they are being studied, or even the comprehension of the procedure indicated by a certain symbol. In this example, if the student does not see that the same symbolism represents the product of that process (Davis, Tall &Thomas, 1997), in this case the volume itself, he or she will not be able to comprehend what is written. Without the "amalgam of process and concept...(called) procept" (Gray & Tall, 1991) the students, through their lack of reading comprehension, demonstrated a lack of flexibility to be able to move between the process and the concept represented by the integral symbol and, in consequence, the inability to respond fluently to this task. On the other hand, higher levels of reading comprehension correlated to the flexibility component of mathematical fluency, or lack of it. When the student possessed reading comprehension (for example, in the problem described above, where the students should distinguish between the evaluation of the integral as a number or a function), he or she was able show flexibility and respond to a set up that they did not remember having seen before. This has a very interesting parallel with foreign language learning itself. It is not unusual for the foreign language learner, who possesses a certain cultural level, to gain a very high level of reading comprehension, which provides him or her with an advantage in terms of flexibility when faced with new contexts while using the language.

Reading comprehension is also related to the compression of mathematical ideas into "thinkable concepts that can be held in the focus of attention (Tall, 2005, p. 1). This aspect is closely related to foreign language reading comprehension as, if structural components have not been compressed, it is impossible to maintain isolated rules, vocabulary, and the like, in the focus of attention, while trying to decipher the material to be read.

Students' written mathematics were analyzed in terms of logical structure, method of attacking the problems and their sketches. An example of mathematical fluency as expressed by writing in relation to, say, the set up of an integral representing volume, given the boundaries of the region that generates it, would be as follows (although this logical sequence is not necessarily unique). The student might graph the curves that form the boundaries, making a sketch of the region, then decide upon the method to generate the volume, depending on the defining equations and the axis of revolution, and finally set up the integral. On the other hand, even when the student might follow this sequence, there may be other problems that present themselves along the way, which will reflect degrees of mathematical fluency. An example in which students were asked to set up the integral which would represent the volume generated by the region bounded by the following equations: $y = x^2, x = y^2$; about $\mathrm{x} = -1$, showed some interesting results for these conclusions. When confronted with the functions "in terms of x" or "in terms of y", the writing parameter evidenced problems coming from mistaken schemas that translated to badly constructed mental models. Of course, as was mentioned above, the parameters interact and the division made for the sake of analysis is artificial as, for example, reading comprehension (or lack of it) plays an important role in the problems that are shown through the writing parameter. The procept concept helps to explain that when the functions are presented as $y = x^2, x = y^2$, it is not always apparent to the student who still struggles with the function as a concept, that $y = f(x)$ and $x = g(y)$. Additionally, the concept of inverse function, which should have played a role in the set up of the region by getting the functions in terms of x or y, was not only not used, but was seen as "adding to the confusion" when suggested to the students who were having problems with the set up. Students' writing shows several methods of attacking the problem. One such method was just graphing the two functions separately, which of course led to a dead end, and another was writing in terms of y and graphing in terms of x. The writing parameter also reflects the flexibility component of mathematical fluency, as shown when futile approaches are discarded and others tried out, or when there is an insistence on sticking to strategies that have shown will not give results. However, the higher levels of written mathematics in terms of logical structure, method of attacking the problem and sequence of steps were seen to correlate to the efficiency component of mathematical fluency, that is, the existence of schemas that enable the development of strategies, translating into less time and effort in actually solving the task. This can be seen in some examples of the written work of the students (Appendix E).

Another issue related to fluency and reflected in the writing parameter was the line $x=-1$ drawn as the line $y=-1$. This has been a recurrent error in my observation of calculus students, and is also mentioned in Appendix A, where some results of the pilot study are presented. This writing mistake evidences a lack of fluency, in particular in the accuracy component. It points to a cognitive obstacle with a didactical origin, as for several years students are taught that the x-axis is the horizontal axis, and the y-axis is the vertical axis, and the association remains in the students' mind.

When dealing with the parameter of spoken mathematics, and what it exhibits in terms of mathematical fluency, it is important to first distinguish between spoken mathematics used as a technique to promote learning in mathematics, and as a parameter to detect mathematical fluency as used in the present study, where "speaking" mathematics is not considered a method to foster mathematical fluency, but an indicator itself of mathematical fluency. At the same time, it is relevant to mention that the act of speaking has been shown to influence the thought processes themselves. According to Vygotsky, cited by Whittin and Whittin (2000) "talking does not merely reflect thought but it generates new thoughts and new ways to think". In this sense, the speaking parameter reflected mathematical fluency in the flexibility component, because the more fluent students, in the interviews and in the classroom setting, often changed strategies and began to think in a different way, as they were speaking.

In general terms, higher levels of spoken mathematics correlated to the accuracy component of mathematical fluency, and there was a direct relationship between coherent and fluid verbalization of their acts, and accurate responses. When Laura confused parallel and perpendicular verbally, she was also reflecting the mistake in her sketching, and that mistake had dire consequences when choosing the method to use. When Susan said "in terms of y" and proceeded to work in terms of x, the spoken incoherency translated into an inaccurate procedure when she set up the integral in terms of x, with a dy at the end. On the other hand, It has been suggested in the literature that students often develop their own language, in which the standard terminology as used by mathematicians is not always employed, or is confused, but that this is not necessarily indicative of a lack of understanding (Findell, 2001). However, as far as this study shows, the speaking parameter does reflect mathematical fluency and is correlated with conceptual understanding. One very interesting example is the following. The students were asked to explain how the formula $s=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$ was developed. In three of the interviewees and more than half the class a common conceptual mistake was made, which consisted in interpreting the straight line segments which are "summed" under the integral sign to calculate the length of the arc of a particular curve, as "derivatives" (really, tangent lines). This common conceptual mistake was detected by the speaking parameter because their sketches, which would be considered as a writing parameter, correctly illustrated the idea behind the development of the arc length formula. It also was seen that students refer to the "tangent line" and the derivative as if they were the same. In other words, the students did not explain the arc length, which is a distance, through the sum of distances. Of course, when they were asked what it means to sum derivatives, they were at a loss. As was mentioned, this conceptual error would never have been detected without the speaking parameter. Also, through spoken mathematics and the use of extra-mathematical and structural metaphors, it was possible to detect mathematical fluency. Once again the accuracy component of mathematical fluency, especially in the spatial sense, was made manifest when, for example, a student was made aware of her mistake in confusing the line $x=-1$ with the line $y=-1$. Once she saw she would be revolving about a vertical line, she mentioned that it would be like a “mirror” (as if revolving about a horizontal line would not), and that it would be “reflected”. These metaphors, the first one extra-mathematical and the second one structural, were not helpful as, after probing, it was seen that she really did have a mental model of a reflection in the plane.

Spoken mathematics is the most direct way to detect metaphors that are used by students, because in mathematics, unlike foreign or native language (where students do creative writing), metaphors do not usually appear as such in students' writing, although they are present in their mental structures. Metaphors used by students, as has been seen, can be induced by particular teaching methods, their own attempts to understand by referring to previous concepts (allegories), among other origins. The actual names of the disk and shell methods correspond to extra-mathematical metaphors, and it was evident that the “disk” metaphor was much more helpful for the majority of students in the class than “shell”. The structural metaphor that was used instead of “revolving” about a line, was “rotating” about. In one student it was noted that while saying “rotating”, she actually performed a written rotation about the line $x = y$, as is done in geometry.

The listening comprehension parameter was seen to be an indicator of mathematical fluency, reflecting the flexibility component most of all. This was evident in the interviews and in the classroom. The student who shows poor listening comprehension is seen to persevere in unsuccessful strategies, and does not change even when the futility of his or her approach is made evident. There are several examples of this in Chapter V where the results are presented in detail, and poor listening comprehension in these cases shows errors that are repeated with each new problem as well as a lack of response to suggestions and indications. In the initial phase, when a concept or an instruction is being presented orally, poor listening comprehension also translates to a lack of efficiency, as even when the student pays attention, he or she doesn't possess the adequate schemas that would permit the formation of the mental models required to comprehend the material. Often the student is in the allegoric or integration stage with respect to the concept, and cannot make the connections that are necessary to comprehend whatever is being presented or explained.

All the students interviewed belonged in calculus II, and there is no intention to question their actual antecedents. However, it is emphasized that calculus, and calculus II in particular, are subjects that require an amalgam of local fluencies if the student is to achieve mathematical fluency. What the results seem to show is that, even if the students possess local fluency in algebra, trigonometry, geometry, it is not automatic that the amalgam of local fluencies will occur and mathematical fluency will come about. It was seen in the results that, in certain cases, the student literally "falls apart" and seems to forget basic notions that are clear when working within the given context of algebra, trigonometry or geometry. An example of this were the lines $y =$ and $x =$ that, when presented as lines about which the students should revolve a region to generate a solid, became confused in their minds and, consequently, in their writing (sketches). This is a phenomenon that also occurs in foreign language learning. The student often has a good knowledge of, say, the different tenses, and can show fluency through reading comprehension, writing, speaking and listening comprehension on the local level, that is, in this example, of the past, the present, the future or other tenses when taken in isolation. However, when that student needs to amalgamate the different tenses, and go back and forth to show fluency on a global level, he or she often not only doesn't achieve that fluency, but often looses the accuracy that had been shown on the local level.

Applications of the Integral

The three research questions related to applications of the integral were centered about the students' stages of reasoning, key metaphors and the role that cognitive obstacles play when setting up the integral for applications different than finding the area under the curve. The applications of the integral provide a rich setting for the exploration of stages of reasoning in relation to the concepts being introduced and their connection to previous knowledge, as this subject matter embraces a combination of geometrical knowledge and intuition, analytical thinking and a constructive use of algebra that implies decision making. Logical thinking is necessary, but dependent upon already existing local fluencies. The capacity to amalgamate the local fluencies, and form mental models based on appropriate schemas that permit the varied applications in terms of generating volumes and surface areas, calculating arc lengths and physical concepts such as work, is what defines the stage of reasoning. Even when there are formulas for realizing the actual calculations, the reasoning process which leads to the correct use of these formulas is of a qualitatively different nature than the "plugging in" that tends to be related to following formulas. The four interviewees could be classified in terms of the applications. One was at the integration stage, one at the analytical stage when dealing with the set up of the integrals to generate volume, but at the integration stage when having to "work backwards" (see Chapter V) and relate areas and volumes, and two actually at the synthesis stage. The four parameters of foreign language learning were used to arrive at these conclusions, through an analysis of the students' written work, spoken mathematics, reading and listening comprehension, transcriptions of which are presented in Chapter V.

Laura, the student at the integration stage, independently of possessing good local fluency in algebra, was unable to relate the new concept of applications of the integral to her known algebraic and (much weaker) geometric concepts. She was determined to memorize the formulas of, say, the disk and the shell methods, and could explain the difference between them. However, when she was confronted with actual equations defining boundaries of the region, she was unable to relate the region (when she could find it) to those formulas, and even confused basic geometric concepts such as *parallel, perpendicular* and *hypotenuse.*

Susan oscillated between the integration and analytical stage. When she was asked to set up the integral, after asking many questions and quite a bit of trial and error, she could relate her previous knowledge and local fluency in algebra and geometry (her geometric intuition is very good) to the new concept. However, when having to "work backwards" in the sense of sketching the volume which an already "set up" integral represented, she did not show the capacity to relate the new concept of *applications of the integral* to known concepts. In this case, as was mentioned above, the integral symbol was not assimilated as a procept, and was seen exclusively as a process. For this reason she wanted to evaluate the integral, and seemed to possess no reading or listening comprehension, as the instruction explicitly asked for a sketch, and she herself had read the instruction aloud.

The other two interviewees, on the other hand, were clearly at the synthesis stage, as the transcriptions in Chapter V show. They were able to relate the new concepts to known concepts, and use the new concepts to "solve problems, develop strategies...and create allegories". (Knisley, 2000). When asked to make a sketch of the same integral, once as a volume and then as an area, Tom not only had no problem distinguishing, but also coined an extra-mathematical metaphor, referring to "sister functions".

The key metaphors that the students in this study used when dealing with applications of the integral are of a geometric and extra-mathematical nature. They refer to the actual figures, and can be helpful or misleading depending upon the geometric accuracy of the initial figure. The terms "disk" and "shell" are metaphoric themselves, although three out of the four interviewees, and more than three fourths of the class understood the disk metaphor, while the shell metaphor was better understood by the extra-mathematical metaphor, coined in the classroom setting, of "concentric cylindrical cans". However, it was still very difficult for the students to picture revolving about, say, the vertical axis, and setting up the integral in terms of the horizontal axis.

When having to set up functions in "terms of x" or in "terms of y" it is, of course, the concept of inverse function that is at play. It was seen that, as a general tendency in the interviewees and in the class, that students tried to keep the function in terms of x, even when this meant a more difficult set up, an impossible set up, or using the wrong set up (usually disk method instead of shell method). As was mentioned above, the procept which would have permitted students to realize that $y = f(x)$ and $x = g(y)$ as functions had not been developed in the majority of students, even though functions and inverse functions are a basic constituent of algebra. The usual extra-mathematical metaphors for inverse function, such as “undoing the first function”, “canceling”, when I used them, did not really seem to be helpful, and pointing out that we were working with inverse functions, far from being seen as a useful tool to clear up confusion, was seen as adding to the confusion.

It was hypothesized in this study that, as the definite integral is introduced in the first calculus course as representing the area under a curve, and the whole process of Riemann sums is motivated by areas of rectangles, there could be a cognitive obstacle that is impeding mathematical fluency when, in the second calculus course, the student is confronted with the variety of applications of the integral, which can represent not only the area under the curve, but volumes, surface areas, arc lengths, and physical concepts such as work, moments and fluid force. At the same time, the TI-83 was the calculator included in the syllabus (this was a college policy, decided by the math department, and not individual), and was used in class demonstrations with the projector. When students set up the integral to be revolved about a line, and used the graphing feature of the calculator to compute the numerical value (which was done even when the students had to compute the integral manually, as they would check the answer), what they saw on the screen was the area under the “new” curve. This was mentioned in class, and those students possessing calculators with a three dimensional display (for example, TI-89) were encouraged to download software which actually generates the volume (which no student did). However, in the case of the interviewees there did not appear to be a cognitive obstacle in terms of applying the integral to obtain numbers which would have a meaning different from the area under the curve if units were to be attached. Nonetheless, I still think that this should be studied further, and if the hypothesized cognitive obstacle could not be detected in the interviewees under the methodology used, it could be due to the following explanation. The experience that students had with attaching units to the integral was nonexistent. Up until the section on work, in the class context, no units had been attached to the numerical results of the different set ups. No problems with applications in actual measurements had been assigned, and my talk about one, two and three dimensional applications could have been assimilated mechanically, rhetorically, and repeated by the interviewees. The only parameter for measuring mathematical fluency that would have reflected this obstacle in the interviews was speaking (spoken mathematics), as the writing parameter did not call for units, and poor reading or listening comprehension, in terms of understanding the differences, would not show in an answer that did not require the attachment of units. When I asked the interviewees about what they thought in terms of the integral being used for calculating one dimensional objects (lengths), as well as areas and surface areas in two dimensions, and volumes in three dimensions, the responses were varied. On one extreme I received the comment transcribed in Chapter V in which Laura expressed “I don’t care what the integral means. I don’t care if it’s an area or a volume, I just want to integrate” or Susan who said “the integral? You always ask “how did this come about?” I usually think ‘someone told me to do it this way’ and I do it.” On the other hand, Tom showed a marked ability to make connections when he said “Because in every case we are summing up the little divisions, you know, the dx ” and Paul’s somewhat philosophical response was “I think it’s important so we can see what’s going on behind the scenes, not just that something magical happens. I enjoy stuff like this, seeing how it works, all the different meanings, how you get to your answer through all that.” However, all these responses reflected the fact that the integral as a measurement had no practical meaning for the students, more than an abstract number.

On the other hand, even though the exam problems were of the same nature, and did not ask for units (see Appendix C), the one student (who was not an interviewee) that attached a measurement to the integral did put area instead of volume when the question specifically called for volume. This student presented the integral as $A = \int$, and the answer as “Area=4” (See Appendix E). Even though this was just one case, it was also the only case in which it seemed to be important to the student to attach a meaning to the numerical answer, and could be indicative of the way in which the definite integral is originally introduced to the student in the first calculus course, as the “area under the curve”. On the other hand, as can be seen in Appendix A, interviewees of the pilot study also expressed volume as the “area under the curve”. Further exploration of this aspect could be very illustrative.

It has been mentioned that, through the speaking parameter, it was detected that students, even when explaining the arc length formula as the sum of straight line segments, confuse these small distances with summing "derivatives" (tangent lines). This aspect needs to be studied further, but I hypothesize the existence of a cognitive obstacle stemming from algebra and trigonometry. When the right triangle is used to explain the slope of a straight line, as well as the distance formula through the Pythagorean Theorem, plus the trigonometric relationships, in particular $\tan\theta$, students do not necessarily form the schemata in the long term memory that would permit them to make the connections and distinctions that the calculus context requires. Even when they apparently possess local fluency at the algebraic and trigonometric levels, the mathematical fluency required at the calculus level is an amalgam, and demands flexibility.

Fundamental Theorem of Calculus

When confronted with the question of identifying the difference between the two integrals $\int_a^b f'(x)dx$ and $g(x) = \int_a^x f'(t)dt$, all the interviewees affirmed to have never seen the second set up, with a variable as a limit. When asked if they remembered the Fundamental Theorem of Calculus, which they had all seen in its two versions, they all admitted to not remembering anything about it, except the name. As the meaning of the two integrals was explored, using the parameters, in particular reading comprehension, it was seen that the first version of the Fundamental Theorem of Calculus, used in evaluating the definite integral, is remembered procedurally, while the second version is seen as completely unfamiliar. However, local fluency in algebra did permit, through reading comprehension and writing out a potential process of evaluation, the realization that the second integral represented a function. If the student can understand the integral symbol as representing the process of integration and the product of that process, which in the case of the first integral above is a number, and in the second a function, he or she has developed the flexibility explained by the procept. In three cases, under quite a bit of prodding and explanation in the interview, the students also could express this function as representing different cut-off points of the sum, and it was finally explained that the integral represents rate of accumulation, much as the derivative represents rate of change. However, there is no guarantee that these concepts will remain with the students, even when they seemed to have arrived at the analytical stage, and it is more than evident that the first introduction to the Fundamental Theorem of Calculus had no impact in terms of understanding the notion of accumulation.

The title of this study implies that underlying notions related to integration, such as accumulation, are connected to mathematical fluency. If this is so, how is it that none of the students, even the ones with a high level of performance, remembered anything about the Fundamental Theorem of Calculus more than its name. To be coherent with the theory developed around mathematical fluency, it is important to mention that as defined in this study, mathematical fluency is more than performance. It also has conceptual understanding built in to the flexibility component. The fact that certain students' performance was accurate and efficient, does not classify them as totally fluent under the criteria of this study. The flexibility component of mathematical fluency is especially important when it comes to problem solving and modeling, and it is with respect to these aspects that conceptual understanding plays an essential role. These aspects were not part of the interview questions, given that the questions themselves were based on the material covered in the standard syllabus of the traditional second calculus course, over which I did not exercise control in terms of changing the approach to more conceptual and model oriented. However, further study can and should be done with respect to the importance of understanding the underlying notions of integration, both in terms of performance in the second calculus course with a more reform oriented approach, and in the context of the following courses the students will take, whether these be in mathematics or in their areas of specialization, and in which they will need to apply and model using the integral.

The first version of the Fundamental Theorem of Calculus, without its name, is used as an algorithm, and the students did not relate the integral as a concept to previous concepts such as the derivative, or even multiplication, showing themselves to be in the allegoric stage in terms of understanding this particular aspect. The proof of the Fundamental Theorem of Calculus, included in the book that all of them used in calculus I, is based on algebraic manipulation with telescoping sums so that, even if it is consulted, it does not really project the idea of accumulation and change. The result of this study seems to be coherent with the literature in terms of the maturity level needed to understand the Fundamental Theorem of Calculus, and it has been questioned (Thompson 1994, Knisley 2000) if the way and the moment it is introduced in the standard calculus I curriculum are adequate. Another aspect, which was mentioned by one of the interviewees, Paul, is that the Fundamental Theorem of Calculus, with such an important name, is introduced at the very end of calculus I, in one or two class sessions, and then never referred to again in the calculus sequence. As he mentioned in the transcription presented in Chapter V, "...it's a good thing to talk about every now and then, not just once or twice and drill it in someone's head, but maybe go back over it again at points in calculus I, II and III. That would make people understand it a little more, and understand what we're doing." Perhaps if the Fundamental Theorem of Calculus was revisited with any new applications, to reinforce the relationship between derivative and integral in the particular case, it would be possible for calculus students to retain the fundamental notions, and improve their mathematical fluency when confronted with calculus "down the road". It also might reinforce the continuity in terms of the applications that are introduced in the calculus II course, how they relate to the original presentations of the integral, and how the integral not only can be used to calculate and represent definite areas, volumes, arc lengths and physical quantities as well as families of functions (antiderivatives), but can also be used to calculate particular functions, where the rate of accumulation is important.

A Comment on Mathematical Fluency, Fluency in Foreign Language and Types of Mistakes

Before proceeding to the section on implications, I would like to talk a little about mathematical fluency and fluency in foreign language, in terms of the type of mistake that is made. The native or fluent foreign language speaker makes mistakes all the time, but the type of mistake that is made is very different from the mistakes made by the non-fluent speaker. In the case of foreign language, for example, certain grammatical mistakes or confusing gender, are indicative of the non-fluent speaker, while, say, other grammatical mistakes or confusing certain words, are not, and can be made by the fluent or native speaker, without any question of their mastery of the language. A similar phenomenon occurs in terms of mathematical fluency. There are certain mistakes which represent deep conceptual misconceptions, cognitive obstacles, erroneous or weak schemas, and lack of local fluency in terms of prerequisites, and others which just reflect the lack of concentration at the moment, but do not imply lack of mastery or important misconceptions. In the case of the present study, this kind of difference is exemplified in the results. An example would be the difference between not being able to use the concept of inverse function to set up 1b, which reflects a lack of fluency, and an algebraic mistake in terms of the intersection points of two curves, to find the region, which is more of a "slip".

Implications

The implications for future research are multifold; on the one hand, the use of the four parameters of foreign language learning: reading comprehension, writing, speaking and listening comprehension to indicate mathematical fluency as defined in this study, is not by any means limited to research in calculus learning, and can be employed at any mathematical level. It is a tool in doing research in mathematics education at the pre-school and school levels, as much as it is when doing research in collegiate mathematics education. It depends on the researcher(s), their expertise in the symbolic language and mathematical content at the level they are researching, as well as their familiarity with the local fluencies that form the background of the particular area and level of mathematical fluency that they are investigating. The present study was totally qualitative, using interview, action research and observation techniques, but that does not mean that the construct of mathematical fluency as measured by the parameters of foreign language learning does not lend itself to quantitative techniques. Testing for fluency in foreign language is often on a massive level (the Test of English as a Foreign Language, TOEFL, for example), but the parameters of reading comprehension, writing, listening comprehension, and recently, speaking, are part of the evaluation process. This means that testing for fluency in mathematics, defined with the components of accuracy, efficiency and flexibility, this last component including conceptual understanding, can also be designed on a level in which quantitative techniques, descriptive and inferential, can be used. These designs can be for purposes of research, or also in terms of placement or qualifying exams. The actual logistics for designing research or examination based on mathematical fluency and the four parameters are related to the mathematical content of the specific area being researched or evaluated.

On the other hand, teaching strategies can take into account the concept of mathematical fluency as defined in this study, and use the four parameters to measure the process. Evidently, much of this is already being done on individual, group and institutional levels, and what this study tries to do is offer a means of systematizing, making precise and homogenizing by naming, a theoretical and practical approach.

The implications of this study for further research in the particular aspect of calculus learning which consists of applications and conceptual understanding of the integral, and their relation to mathematical fluency, have been evidenced in the results and conclusions. The problems in making connections with previous knowledge that students show, their inability to achieve mathematical fluency in calculus even when their local fluencies in algebra, trigonometry, geometry, are adequate, the tendency towards memorizing formulas instead of following the geometric development (disk and shell methods), are all questions that need to be further investigated. The absence of the Fundamental Theorem of Calculus from any reference frame when applying the integral shows the necessity of reestablishing, or replanting, the introduction and continuity of its importance and use, both operational and conceptual. Conceptual understanding of the integral and its applications needs to be researched, as well as the way it is being introduced and assimilated. The results of this study tend to show that students, instead of feeling the need of conceptual understanding to be able to perform, see the introduction of concepts as superfluous. This is also seen to be related to the type of problems they are presented with, and how these are connected to any kind of modeling, or even the essence of the applications themselves (for example, entire sessions are spent showing how to generate volumes, surface areas, and arc lengths with the integrals, but units are never attached to the numerical answers that are found).

The study also detected aspects in the precalculus curriculum which are consistently shown to fail when they are needed in the context of calculus, corroborating other studies, some of which are mentioned in the literature review on learning in calculus, Chapter III of this study. Some stem from direct algebraic and trigonometric rules and relationships (such as the trigonometric relationships and identities), while others are of a more conceptual nature, and can be explained with the terms of cognitive obstacles (such as confusing the lines $x =$ and $y =$, the inverse function, or confusing distance and derivative).

The present study can be considered, in more than one way, as seminal, given that the construct of mathematical fluency as a global amalgam of local fluencies, which can be measured by the four parameters of foreign language learning, needs to be developed and tested in different contexts and manners, as has been mentioned. The measurement of foreign language fluency with the four parameters has a considerable history, and instruments have been developed so that these measurements are precise, which is what must be done in the future with the construct of mathematical fluency. The instruments to be developed, using the parameters of speaking, writing, reading comprehension and listening comprehension, must be adapted to the mathematical subject in which fluency is being measured. Care must be taken to distinguish local fluency in prerequisites, in designing the use of the four parameters, as well as in the definition of the measurement criteria itself.

On the other hand, in terms of research, many more studies must be carried out using the model of mathematical fluency as measured by the four parameters of foreign language learning before its general utility can be affirmed; as was mentioned above, these studies do not necessarily have to be restricted to calculus learning, or even collegiate mathematics. It is hypothesized, on the basis of this study, that using the four parameters to measure mathematical fluency can be very important in detecting the local fluencies essential to new concepts. It is also hypothesized that the parameters can help to understand how the student amalgamates, or fails to amalgamate, their local fluency if they possess it, into mathematical fluency in the new concept. The model needs to be repeated and perfected by researchers in any area of mathematics education. It is also understood and acknowledged that much of the qualitative research in mathematics education has relied on the use of some or all four of the parameters. For example, Ferrini-Mundy and Graham (1994) explained that their "interviews were designed around a series of tasks developed by the researchers" in which "Students were asked to complete the tasks and to 'think aloud' as they did so" (p. 33). The speaking parameter is used in this case to measure the results; the results themselves could also be presented in terms of mathematical fluency. It is important to reiterate that the intention of the part of this study dedicated to development of the construct of mathematical fluency, and the use of the four parameters, is to help systematize and standardize categories and techniques used in mathematics education research.

Calculus is often considered as the gateway to advanced mathematical thinking. If students can achieve mathematical fluency in calculus, that gateway will lead to further success in terms of mathematical thinking and performance. The goal of any researcher in mathematics education is to look for ways to understand mathematical thinking, communicate that understanding, and offer alternatives to improve mathematical teaching and learning. The author of the present study hopes to have contributed in some way to this ongoing process.

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APPENDIX A

Results and Conclusions of the Pilot Study

Three students from a community college in the Northeastern United States, who had taken Calculus II during the Spring semester of 2000 with me, were interviewed. I also knew their performance on quizzes, exams, as well as their class participation, and one had come regularly to get help during my office hours.

General Procedure

As was stated in the measurement criteria for mathematical fluency, to detect the comprehension of written mathematics (reading comprehension), students were given written problems and asked to read silently, read aloud, and explain what they understood. As all sessions were recorded, their explanations were later coded for coherence and precision, using as guides their extra-mathematical and structural metaphors, the use of terms (mistaken, incomplete) related to the subject, and their descriptions of their strategies, their procedures and their concepts. Their written work was analyzed to detect logical structure, strategies of attacking the problem, sketches and graphical representations, and sequence and order of steps. To detect the comprehension of spoken mathematics (listening comprehension), I explained in mathematical terms, at the level of the calculus II course (keeping in mind the concept of local fluencies as the components of mathematical fluency at a particular level), some process or concept related to what they were working on, and then asked them what they had understood.

When the students were asked to "set up the integrals that let you find the volume of the solid generated by the region bounded by the graphs of the equations, about the indicated line" (this is standard language, with minimal variation, in Larson and Stewart, taken as prototypical standard calculus texts), for question 1c, all three students, in the three different interviews, started off with the same task, that is, graphing the function $\sin x$. However there was a difference in the procedures. Nathan graphed on his calculator, while Bill and Ben graphed by hand. Still, there was really not a big difference in terms of the initial "attack"; All three looked for a general graph, and then limited the domain to the interval $(2\pi, 3\pi)$. Nathan, who is definitely the weakest student of the three, spent more than double the time of Bill to just find the graph on "some" domain. He did not go back to his calculator to check between 2π and 3π, but graphed by hand, ignoring the graph he had obtained on the calculator. After graphing by hand the general graph of $\sin x$, he decided to proceed as follows.

> **Nathan:** "Now I write down the parameters that you want me to do, $y=0,\ x=2\pi,\ x=3\pi.$" From looking at this I know that every time y gets to be a π, like 1π, 2π, 3π, then y is 0, so you're asking me if I can kind of draw this." "1π, 2π, 3π, and I know that at every π is 0, and this is $\sin x$, so it only oscillates between 1 and -1."

At the same time Nathan was confusing x and y in his spoken math, the graphing that he was doing was fine. This seems to be a typical pattern in mathematics students. In other words, his *image scheme* (Sfard) was correct, but his "translation" to spoken language was incoherent.

After drawing the graph, the three students proceeded to make a sketch, as far as possible, of what the three dimensional solid of revolution should look like. However, Nathan was not clear about what the "region bounded by the graphs of the equations" really meant.

> **Nathan:** So you want me to bound the graph on the y-axis. On the demonstration graph I drew a longer line, because now I need to bound the graph on the y-axis, which is just going to be a flip."
>
> **Mariana:** We're just bounding the region, the region is the area that we're going to "flip". Is this the region that you want to revolve?
>
> **Nathan:** Yes
>
> **Mariana:** So it is bounded. It's bounded by this line, what is the equation of this line?
>
> **Nathan:** The x-axis, which is 2π and 3π. (*Nathan verbally confused the x-axis itself with the endpoints on the x-axis)*
>
> **Mariana:** The equations are there (*showing the equations written on the whiteboard*) ... it's bounded by $x = 2\pi,\ y = 3\pi,\ \mathrm{y} = 0$ and $y = \sin x$.
> **Nathan:** So it's just a flip.

In this weaker student, the process of bounding a region by means of the given equations could not be carried out. The mathematical fluency parameters showed that the student could not relate the "given equations", as functions, to his extra-mathematical notion of boundary. Bounded sounded to him like volume, and the volume is conceived of as a flip. If we are sincere, we also realize that at this stage of the calculus program, students have never really encountered a formal definition of *boundary*! At this point, it seems that Nathan is somewhere between the stages of allegorization and integration, when faced with the concept as "generating solids of revolution by revolving a region about a line". Also, the metaphors he has created are not helpful.

On the other hand, Bill bounded the area immediately, and proceeded to decided what the volume generated would look like. Ben, on the other hand had no problem with the region, but could not really see the solid, and didn't think it was important.

> **Mariana:** What would that solid look like?
> **Ben:** I'm not sure, but I think I can set up the integral.
>
> When I asked Nathan what he thought the solid would look like,
>
> **Mariana:** What would you call the solid?
> **Nathan:** disk.

Here it is obvious that he was confusing the solid itself with the method that he had decided to employ.

In terms of the method, the three of them decided they would use disk or washer. They all decided without analyzing first that using the washer method would signify revolving around the y-axis, and that they would have to deal with the integral of the function $\arcsin^2 y$. When Nathan got to this point, he was also confused as to if it should be $\arcsin^2 x$ or $\arcsin^2 x^2$. This was one of the many evidences of lack of mastery of the previous mathematical concepts necessary to the task at hand, which was manifest in the less fluent students. When Bill tried to set up the integral using the washer method, he ran into a problem. He couldn't find a way to express the big radius and the little radius. At the same time, he didn't realize that, in terms of the y-axis as the independent axis, $\arcsin y$, when taken from 2π to 3π, is not a function. Also, he did not realize that when we talk about big radius minus small radius, we always need two functions, although one can very well be a constant function (horizontal line). However, as he went through this process, he realized that he could not proceed with the washer method.

I asked all three the same question:

> **Mariana:** If we have rectangles perpendicular to the x-axis, how do we generate the solid?

Once I asked this question, for Bill it was obvious what he had to do.

> **Bill:** Oh, I need to use the Shell method.
>
> **Mariana:** But how does that let you revolve about the y-axis?
>
> **Bill:** It's hard to explain, but look....(he draws), it fills up from inside to outside.
>
> **Mariana:** What does the function represent when you use the Shell method?
>
> **Bill:** Oh, height.

I want to mention something now that I will take up again in my conclusions. As far as strategies go, in terms of "plans of attack" there was really not much of a difference among the three subjects. However, while Nathan could not see the way out of a path that was leading to nowhere, Bill was able to quickly see what would work and what would not, and though he couldn't always get to a successful strategy right away, he would not try to make work an approach that was doomed to failure. Ben, on the other hand, got frustrated quicker, and would give up (something that Nathan never does).

> **Ben:** It's impossible to revolve this about the y-axis.
> **Mariana:** Why?

Ben: How can I set up this integral in terms of y?
Mariana: Do you have to?
Ben: I don't see how, oh wait, the other method.

Both Nathan and Ben had trouble explaining (or drawing) how they could set up the integral in terms of x, and revolve it about the y-axis. It seemed to be easier for them to visualize the rotation of the rectangles, generating slender cylinders, than the shells.

The next set up was $y = x^2$, $x = y^2$, about the line $x = -1$. Nathan proceeded to graph the parabola ("that was easy"). He also was going to graph the two functions separately, and did not seem to realize that he had to find the region.

Nathan: $x = y^2$, how do I draw that?

I had seen Nathan draw graphs in terms of y before; it is evident that what he has done is not in his long term memory, and he does not have schemas for the inverse function relationship. At the same time, when it is pointed out, it is understood.

Bill graphed the two functions without defining one specific independent variable. He did not see the region, because of the way he graphed by hand. When I suggested he use the calculator, he saw it. He decided to use the washer method, but then did not express the y as the independent variable; he insisted that:

Bill: It doesn't matter, x is still x, and y is still y.

When I said that the labels didn't matter, he said he knew that (which he does); he also knows what an independent and dependent variable are, but he could not see the need for expressing x in terms of y until he tried to set up the integral.

Bill: I think you might be right; I need to decide which my independent variable is. I had never thought about that before.

I found this comment very surprising, given that Bill had performed at such a high level in the whole calculus II course. When Bill realized that to use the washer method he would have to set up x in terms of y (which is not complicated for him, given my knowledge of his performance), he decided to change to the shell method, to be able to leave y in terms of x.

On the other hand, Nathan used the graphing calculator to find the region, when he was at loss to do it by hand, but he had no problem understanding that he needed to choose an independent variable, and express the two functions in terms of either x or y. However, he could not see the region, because his window was too big. I suggested he limit it, and then he identified the region.

Mariana: That is your region that you want to revolve? Which function is on top?
Nathan: Square root of x.
Mariana: You want to revolve this around...
Nathan: x = -1. So I make a nice straight line here (*he draws y = -1*).

This is a clear example of not being able to synthesize the information and generalize in a reconstructive manner. If the problem was just about lines, he would have proceeded correctly (I tested that). But his mental image mapped onto the line $y=-1$, and the language of linear equations, which he has been manipulating since high school algebra or before, seems not to be mastered. I would compare this to learning a language; there are foreign language speakers who know all the rules separately, can analyze grammar better than an average native speaker, but will make extreme mistakes when really speaking, that no native speaker would make. It is still a "foreign" language. I think that this might be one of the big differences between the successful mathematics student, and the student that is erratic, at times understanding "big" concepts, or solving difficult problems (which Nathan has done), and at times making mistakes with very elementary processes and concepts.

As I had interviewed Nathan before having interviewed Bill, I was aware of this problem. Bill had drawn the vertical line almost immediately once he said "we revolve about the line $x=-1$". I insinuated to Bill that sometimes people have problems with, say, the line $x=-1$ as a vertical line; he responded that he too had initially been confused. However, for me it was imperceptible, as he rapidly drew the vertical line. The fact is that the successful student, even when admitting he had experienced doubt, did not hesitate; he knew he was rotating about a line parallel to the $y-\text{axis}$. This is behavior that identifies fluency, and relates to fluency in foreign language and music. The fluent foreign language speaker, or musician, will not let a temporary ambiguity in the symbol system interfere with the mental model that he is employing at that moment, just as the *native* speaker of a language wouldn't.

With respect to determining arc length, Bill remembered that the approach is to take an infinite sum of straight line segments.

Mariana: Why do we use the integral?
Bill: Because we take the sum of all the little straight lines.

However, he mentioned that he didn't really understand the step

$\lim_{\Delta x \to 0} \sum_{i=1}^{n} \sqrt{1+(f'(x))^2}\,\Delta x$ to $\int_a^b \sqrt{1+(f'(x))}dx$.

As far as the Fundamental Theorem of Calculus, Bill remembered that the derivative means the slope of the tangent line or rate of change.

> **Mariana:** "If we take rate of change as our definition of the derivative, what is the antiderivative?"
> **Bill:** Won't it be... the change itself?

When I asked him, he said that he had used the word change mechanically, and began to make an analogy with the velocity and position functions. I asked him to think of a key word to represent that change as the right hand side of the interval varied.

> **Bill:** It's like... the sum of changes.
> **Mariana:** An accumulation of whatever that integral is representing.
> **Bill:** Yes, an accumulation.

After that, Bill used the word accumulation when presented with the question.

As far as the question related to the Fundamental Theorem of Calculus, it was possible to detect the relationship with symbols that was inherent in the set-up of the question. Once again, the question was worded in the following way:

> The derivative can represent a rate of change. If a function represents a rate of change, it is the derivative of another function. Let f' represent the rate of change function. What does $g(x)=\int_a^x f'(t)dt$ represent?
>
> What does $\int_a^b f'(x)dx$ represent?
> What is the difference between the two integrals?
>
> The weaker students used incoherent metaphors and confused facts. For example, the first of the above integrals was "explained" in terms of the variables x and t as independent and dependent, and the equation as

representing “change” from area to volume. This could also be related to the fact that the student knew I expected certain terminology, and that “independent and dependent variables”, and “change”, are part of the jargon.

Preliminary Conclusions

There were several preliminary conclusions that I could draw from the pilot study. The students whom I interviewed ran a gamut in terms of their flexibility in moving between the actual process of the task, and the concept that must be mentally manipulated. This flexibility appeared to be related to various conditions that I will identify as:

1) Mastery of the previous mathematical concepts necessary to the task;
2) Possession of strategies in the long term memory (schema), that can be called upon to construct mental models;
3) The use of appropriate (helpful) metaphors as opposed to unhelpful ones.
4) The stage of reasoning.

I will then show how these aspects, measured through the parameters of foreign language learning, can contribute to the understanding of students' mathematical fluency.

When I refer to mastery of previous concepts, I am not referring to just knowing the rules. All the subjects that I interviewed, when asked to express correct procedures in isolation, could do that; I did not interview students who “should not have gotten” to calculus II, and all of them have had some degree of success in mathematics in comparison to the majority, especially at the community college. However, there was a marked difference between students.

The high achiever constructed mental models that adopted a clear path of action, discarding an idea when it was obvious that it would not work. The lower achievers seemed to have no direction when addressing the problem; this also related to the fact that the lower achievers called on much less of the possible algebraic and geometric schemas than the high achiever. The preliminary results seem to support an argument given by Chinnappan (1998), that

> accessing and use of geometric knowledge (in the case of his study) during problem solving is influenced by the organizational quality of that knowledge. It appears that students who structure their prior geometrical knowledge into chunks or schemas also develop an understanding of when and how to deploy that knowledge productively during problem solving. (p. 213)

Apart from the obvious mistakes that the student who has not interiorized and encapsulated $x = -1$ will make, this type of lack of clarity will influence the consequent chain of decisions in terms of the method to choose (disk, washer or shell) and, most important, the actual solid that results. If I am thinking about the nature of my students, where the majority are going into applied areas, it doesn't make any sense at all to count as valid a problem, even though it is correctly carried out once it is set up, that has been set up wrong from the start.

The presence of cognitive obstacles can be deduced comparing the mistakes of the students with the literature, but the only cognitive obstacle that was explicitly investigated in the interviews was related to the initial presentation of the integral as the area under the curve, and if this initial presentation affects the future applications. In all but one case, this mistake occurred when I explicitly asked them to derive the formula using the geometric motivation. Once the rectangles were drawn, even though they knew that they would be generating volumes, they said that the integral represented "the area under the curve".

In terms of mathematical fluency, I was careful in the interviews to present the materials so that I could make a reasonable measure of this aspect according to what I was looking for. Efficiency, accuracy and flexibility were measured in each of the cases by using the "foreign language" parameters of writing, speaking and understanding through reading comprehension and listening comprehension. This is why, in the beginning, I wrote down the problem and let the students interpret it. In all cases the students read the problem aloud, and then proceeded to work. As I commented in the section on results, all the students began in the exact same way, graphing what they saw as the function(s) among the equations that were given. The aspect of fluency became noticeable almost immediately. The awkwardness and slow speed, as well as the rigidity (in terms of clinging to unsuccessful strategies) with which the weakest student proceeded, betrayed a lack of understanding, and the constant spoken mistakes (confusing domain with points, dependent variable with independent variable), even when his visual images – sketches– were correct, is indicative of a non-fluent mathematics speaker. As mathematical fluency is defined as an amalgam of local fluencies, it was possible to detect which local fluencies were failing (basic algebra, trigonometry, differentiation) and, for that reason, interfering with the mathematical fluency at the calculus II level. It is important to remember that local fluencies are not disjoint, and there are intersections; differentiation, for example, is an amalgam itself of local fluencies, and we must always talk of mathematical fluency at a particular level. On the other hand, the highest achiever, even when admitting to me that he had experienced a slight confusion –insecurity- when seeing *x=-1*, by no means showed it, or hesitated; the knowledge that he was going to revolve about a line parallel to the y-axis (he had made the sketch) was stronger.

Another aspect that I found is that the incoherencies in spoken mathematics reveal conceptual flaws, as well as the use of terms and relationships that, while meaningless in the actual context, draw from the mathematical language that they have been exposed to, fundamentally in the spoken mathematics they hear in the classroom and I identify this as indicative of a non-fluent state. For example, when I asked the same question about $g(x)=\int_a^x f'(t)dt$ to the weaker students, I got an answer from one of them in terms of the variables $\mathrm{x}\ \mathrm{and}\ t$, as independent and dependent variables, and the equation as representing a change from area to volume (although the word change was there!).

Some of the mistakes that reveal non-fluency which, in turn, reveals conceptual misunderstanding, were presented in the section on results. There was a correlation between non-fluency and the lack of ability to combine known algebraic and geometric facts. In the same vein, poor spatial reasoning and problems with visualization, that were detected through the parameters of mathematical fluency, were related to efficiency, specifically in terms of the applications. The notorious problem with derivatives of trigonometric functions, which then carries over to integration of trigonometric functions, appears to be related to the way in which they are introduced. Only one student could relate the derivatives of the trigonometric functions to the slopes of the tangent lines of the functions themselves, in which case he possessed a schema for working purposes, whereas the others could only resort to their memory, which failed often as not.

In terms of stages of student reasoning and performance, the weakest student was at the *integration* stage (comparison, measurement and exploration to distinguish the new concept from known concepts). He realized that the concept (in particular, using integrals to generate solids) was new, but could not relate it to familiar concepts to be able to perform fluently. He spent a lot of time "setting up" the scenario, usually without actually arriving at the correct set-up, and his work was inefficient. The student at the *analysis* stage was thinking critically about the new concept. He was always waiting for new information I could give, and would then proceed to perform successfully. For example, when he was stuck in terms of using the Washer method, it was enough for me to ask "do you *have* to use that method", for him to ask some relevant questions about the alternative Shell method, and move on.

As far as the use of structural and extra-mathematical metaphors, with respect to applications, the metaphors were mainly extra-mathematical, such as *flip, washer* (but not "shells"), or forcing the figure of the solid to a preconceived idea of what it "should be like".

As far as the relation between fluency parameters and performance, at this stage, the written and spoken (as recorded) results, which are also used to determine reading and listening comprehension, lead me to conclude that:

1) Higher levels of written mathematics correlate to efficiency;
2) Higher levels of spoken mathematics correlate to accuracy;
3) Fluency (as measured) in spoken mathematics is highly correlated with conceptual understanding;
4) Higher levels of listening comprehension correlate to flexibility.

The implications of this study are multifold. On the one hand, I think that other studies dealing with different stages of mathematical learning, whether elementary, intermediate or advanced, would be enhanced if they take into account the concept of mathematical fluency, in the extended sense that I use here. In terms of calculus learning, and calculus teaching, the aspects that have been identified as problem-provoking, whether dealing with precalculus concepts, anterior calculus concepts (from calculus 1), or the new concepts that are being introduced are more than procedural, and can be understood in terms of fluency. On the other hand, teaching strategies can take into account the four parameters of mathematical and placement exams should also consider using the four parameters of mathematical fluency. Also, certain authors who have alternative proposals to, say, the Riemann sum as a first motivation for studying integrals (Czarsnocha & Prabhu, Cordero) can find elements to study their claims.

APPENDIX B

Syllabus

_____________ Community College

MAT252 - Calculus II Fall Semester 2005

Instructor: Mariana Montiel **Office Location: Mariana**
P. L. **P.L.**

Office Hours: Mariana: MW 10-11, F1-2 **Office Phone:**
T 11-12
P.L.: MW 9-10, TTh 11-12

E-Mail Addresses:

Cancellation Announcements:

NOTE: Appointments may be arranged by talking with us before or after class meetings, by telephone, or by e-mail. We are here to help you with this course to make this an enjoyable and worthwhile experience.

COURSE DESCRIPTION:
4 credits; 4 lecture hours
Proficiency Requirements: College Reading
Prerequisites: C or better in MAT 251 or by permission of instructor
Electives: Satisfies Liberal Arts, Free, Mathematics
Description: The second course in a 3-semester sequence. This course is a continuation of MAT251 Calculus I. Topics include: applications of integration including, area and volume, techniques of integration, improper integrals and power series. A graphing calculator is required.

COURSE OBJECTIVES:
Upon successful completion of this course, the student should be able to express her/himself clearly and precisely using mathematical vocabulary and should be able to:

1. use simple differential equations to model real world problems
2. use disk and shell methods to set up integrals yielding volumes of revolution, arc length, and surface area
3. integrate a wider variety of functions using integration by parts
4. integrate powers of sine, cosine, secant and tangent functions

5. integrate by trigonometric substitution and partial fractions,
6. test for convergence and divergence of infinite sequences and series
7. use Taylor and Maclaurin series to approximate functions by polynomials
8. use polar coordinates and parametric equations to explore concepts in analytic geometry

REQUIRED MATERIALS:

Textbook: Calculus of a Single Variable- Early Transcendental Functions; 3th Edition, by Larson, Hostetler, Edwards. Houghton Mifflin Company, 2003. Also highly recommended is the Student Resource Manual associated with the textbook.

Calculator: All students will be required to have a graphing calculator. In the classroom and in your textbook there will be specific instructions with respect to the use of the TI-83 Plus graphing calculator. If you have another type of graphing calculator, you will be responsible for its use in performing computational analysis. Make sure you have the user's manual. See instructor for suggestions. A very limited number of graphing calculators are available for use at the college. .

ATTENDANCE:

Attendance is mandatory. Students are expected to attend each meeting of each class in which they are enrolled. The instructor reserves the right to assign a grade of "NP" to anyone missing more than three hours of lecture or to any student who is unlikely to pass due to missed exams or assignments. The class instructor has full and final authority to decide whether a student is permitted to make up work missed through absence, and on what terms. If a student is absent from class, he/she is responsible for the material covered, as well as any announcements made at that time (Please see the Student Handbook).

STUDENT RESPONSIBILITIES:

All students are expected to take an active role in their learning. Notes should be taken in every class and studied before assigned problems are attempted. Homework, though not necessarily collected, should be done immediately after each class session. For every hour spent in the classroom, two hours of homework/studying are expected. Difficulties with a particular assignment should be taken care of before the next scheduled class session. Our goal is to help you have a successful semester and your active participation is a necessary step towards achieving that goal.

METHODS OF INSTRUCTION: Class will generally begin with questions. An overview of the new material will be then be given. The students will actively participate in the development of the new material. Then the students will be given problems similar to the homework.
Methods of instruction will include lecture, cooperative learning, use of graphing calculators, and peer sharing.

ACADEMIC ETHICS AND PLAGIARISM:

The college expects all students to maintain high standards of academic honesty and integrity. Plagiarism in any form is not to be tolerated. Plagiarism is defined by the college to be the use of any person's work or ideas as though the work or ideas were your own, without giving the appropriate credit (Please see the Student Handbook). Any student found in violation of this policy may be given an F for the course.

GRADING:

There will be 3 hour exams, and a comprehensive final examination. We do not normally give make up exams. We will drop the lowest of the three in class hour exams. If you miss one test, the grade will be dropped. Your final grade will be the simple average of the remaining 2 test grades and the Final Exam grade.

Grading System

Grade	Quality Points	Numeric Range
A	4.00	93-100
A-	3.70	90-92
B+	3.30	87-89
B	3.00	83-86
B-	2.70	80-82
C+	2.30	77-79
C	2.00	73-76
C-	1.70	70-72
D+	1.30	67-69
D	1.00	60-66
F	0.00	≤ 59

The following grades do not affect a student's grade average:

W Withdrawal from course by student within Period 2 (please see Academic Catalog and Calendar)

NW Student is withdrawn (NP'd) by instructor (please see Academic Catalog and Calendar)

FW Withdrawal from course by student (please see Academic Catalog and Calendar)

If there is a student in this class who has needs because of a learning disability or is Deaf or Hard of Hearing, please feel free to come to discuss this with us and/or directly contact the appropriate office below:

- <u>Learning Accommodations Center, 3 (Student Center),</u>
  Serving students with physical disabilities, ADHD, learning disabilities, brain injury, blind or low vision and also psychiatric disabilities (through the Supported Education Program).
- **Deaf and Hard of Hearing Services**
-
- <u>DELAYED OPENING/LATE START</u>:

If a delayed opening is announced over the television or radio, the classes scheduled before the delayed start time are cancelled. Classes beginning after the start time are held. Please contact the School Announcements number at for further information.

Topical Course Outline and Assignment Schedule

Date	SECTION	TOPICS	ASSIGNED PROBLEMS
Sept 7	6.1	Area of a Region Between Two Curves	p. 418: #1,5,7,13,17,19,23,33,37,43
9	6.2	The Disk Method	p. 428: #1,5,9,13,
12	6.2	The Disk Method	p. 428: #17,21,25
14	6.3	The Shell Method	p. 437 #1,3,9,13,17,
14	6.3	The Shell Method	p. 437: #21.23.33.37
16	4.9	Hyperbolic Functions	p. 365: # 7,13,15,21,29,33,37,39,79
19		Problem Session	
21	6.4	Arc Length	p. 447: # 1,5,13,23(a,b,c), 27,29,31
21	6.4	Surfaces of Revolution	p. 447: #33,37,34,45,49
23	6.5	Work	p. 456 # 9, 11, 17, 19, 21
26	6.6	Moments, Centers of Mass	p. 467 # 7, 13, 15, 17, 33, 35
28	5.1	Presentation; Differential Equations,	p. 379: #1,5,13,21,25,
28	5.1, 5.2	Differential Equations:	p. 379: # 29, 41, 65; p.392: # 19,27
30	5.2	Separation of Variables	p. 392 #43,47,57,61,63
Oct 3		Review for Test 1	
5		**TEST # 1**	
5	7.1	Integration Techniques	p. 486: # 15,19,23,27,35,43
7	7.2	Integration by Parts	p. 494: # 11- 35 every other odd
12	7.2	Integration by Parts	p. 494: # 47-57 odds
12	7.3	Trigonometric Integrals	p. 503: #3-37 every other odd; 47,49
14	7.4	Trigonometric Substitution	p. 512: #5,9,13,19,23,27
17	7.4	Trigonometric Substitution	p. 512: # 31,41,45,49,61
19	7.5	Partial Fractions	p. 522: # 7,11,17,21
19	7.5	Partial Fractions	p. 522: # 25,29,41,61
21		Problem Session	

24	7.7	Indeterminate Forms and L'Hopital's Rule	p. 537: # 1,5,9,13,17,21,25,31,35,55
26	7.8	Improper Integrals	p. 547: # 1,5,7,11,17,21,25
26	7.8	Improper Integrals	p. 547::# 29, 33,37,81,85,87
28		Review for Test 3	
31		**TEST # 2**	
Nov 2	8.1	Sequences	p. 564: #1,5,9,13,17,21,27,31,35,39
2	8.1,8.2	Sequences; Series and Convergence	p. 564: #57,71,77,81,87; p. 573: 1,5,
4	8.2	Series and Convergence	p. 573: # 33, 39, 45 51, 57
7	8.3	The Integral Test and p-Series	p. 581: # 1,5,9,11,13,17,21
9	8.4	Comparisons of Series	p.587:#3,7,11,15,19,23,29,33
9	8.5	Alternating Series	p.595:#9,13,19,23,29,31,41,45,49
14	8.6	The Ratio and Root Tests	p.603 # 13-31 eoo, #35-41 odds, #43
16	8.7	Taylor Polynomials and Approximations	p. 613 # 1-4, 5,13,19,25,27, 41,45,49
16	8.8	Power Series	p.623 # 1-33 eoo
18		Problem Session	
21	8.9	Representation of Functions by Power Series	p. 630 #1-25 odds
23	8.10	Taylor Series and MacLaurin Series	p.. 641#1,5,19,25,29
28		Review	
30		**EXAM # 3**	

Nov 30	9.2	Plane Curves and Parametric Equations	p. 672 #1,3,9,17,21,33,35
Dec 2	9.3	Parametric Equations and Calculus	p. 681 #1,3,7,9,17
5	9.3	Parametric Equations and Calculus	p. 681 #35, 39.49.51
7	9.4	Polar Coordinates and Polar Graphs	p. 691 #1,5,11,21,25,31,33,55,67,73
7	9.5	Area and Arc Length in Polar Coordinates	p. 700 #1,5,9,13,17,41,51
9		Problem Session	
12		General Review	

Instructor's Note: The instructor reserves the right to make changes to this syllabus at any time during the semester. A new syllabus may or may not be distributed at the discretion of the instructor.

APPENDIX C

EXAM

Name:

Calculus 2

Test 1

Directions: This is a closed book test. You may use a graphing calculator. You must show your work and your work must logically lead to your answer to get full credit. Leave answers with radicals and 'π' where appropriate. If you need more paper, use the back of the pages.

1. Make a sketch of the region enclosed by:

$y = 2\sqrt{x}, x = 4, y = 0$

2. Use cylindrical shells to find the volume of the solid that results when the region enclosed by the equations in problem #1 is rotated around the y-axis.

3. Sketch the region enclosed by:

$y = x^{3/2}, x = 1, y = 0$

4. Use disks to find the volume of the solid that results when the region enclosed by the equations in problem #3 is rotated around the x-axis.

5. Find y' if $3y = 2x^{3/2}$

6. Find the arc length of the curve, $3y = 2x^{3/2}$, from x = 0 to x = 3.

7. Find y' if $y=\sqrt{9-x^2}$

8. Find the area of the surface generated when $y=\sqrt{9-x^2}$ from $x=0$ to $x=2$ is revolved around the x axis.

9. Find the work necessary to move an object from a point 1 foot from the origin to a point 25 feet from the origin if the force is given by:

$$F(x) = \frac{3000}{x^2}$$

10. Find the general solution of the following differential equation:

$$2\frac{dy}{dx} = \frac{3x^2 + 2}{y}$$

11. Find the particular solution to the equation in problem 10 given the following initial conditions: $y = 5$ when $x = 3$.

12. Set up a differential equation that describes the following situation and find the general solution:

The rate of change of a population is proportional to the population. The constant of proportionality is .03.

Bonus: (10 points)

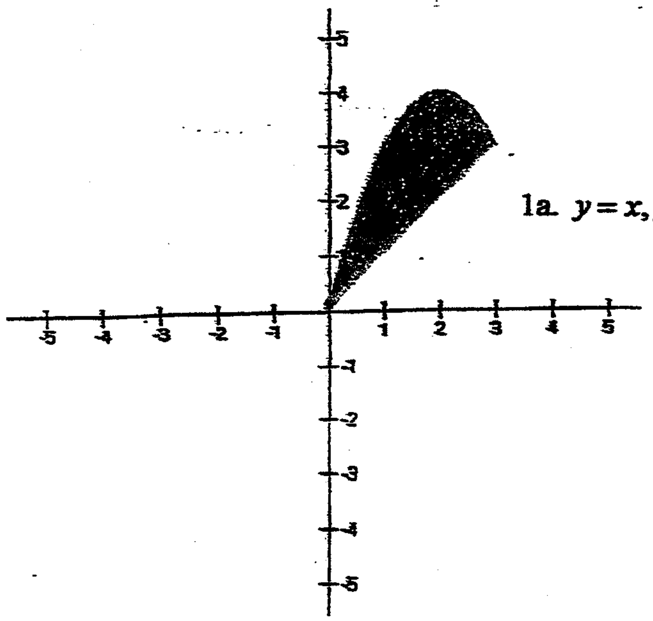
Sketch the graph of the region bounded by:

$$f(x) = x^2, y = 0, x = 0, x = 2$$

Use the disk or shell method to find the volume of the solid formed by revolving the region about the line x = 4.

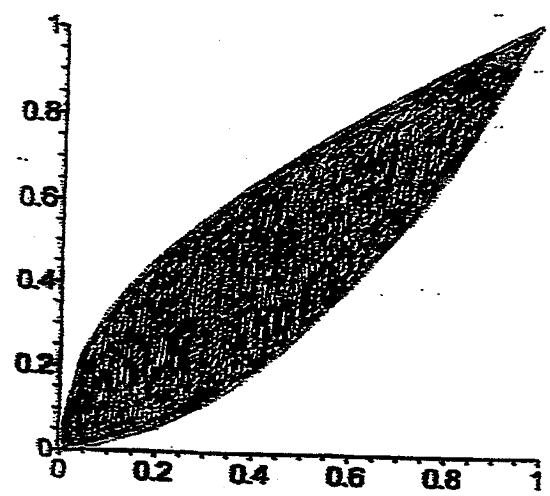
APPENDIX D

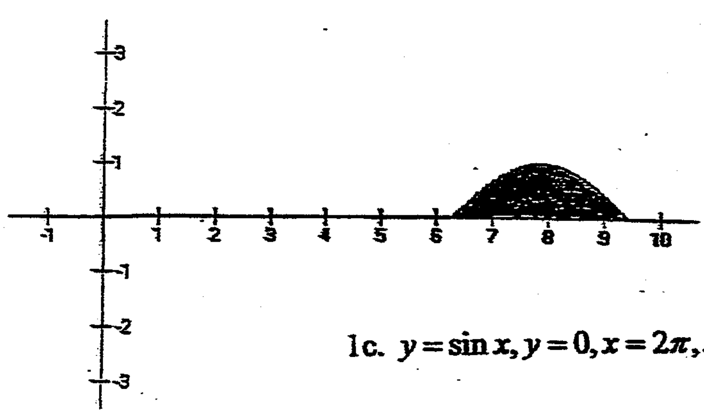
Graphs of the Regions of Questions 1 and 2



1a. $y = x, y = 4x - x^2$; about x-axis.

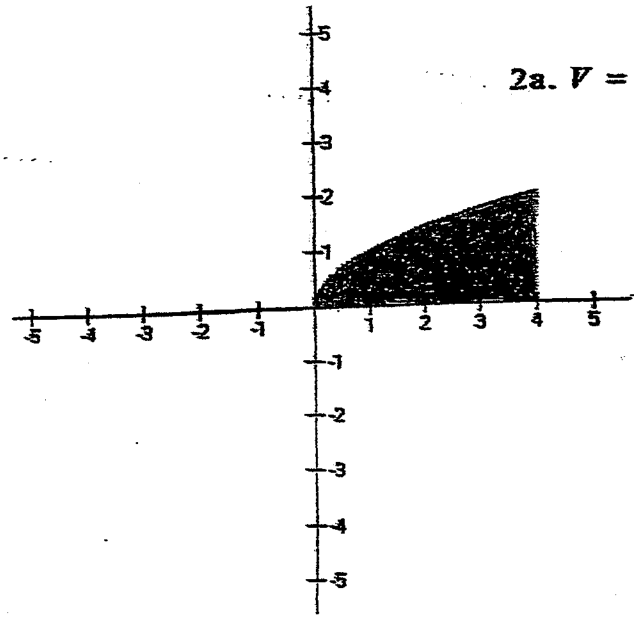
1b. $y = x^2, x = y^2$; about $\mathrm{x} = -1$.



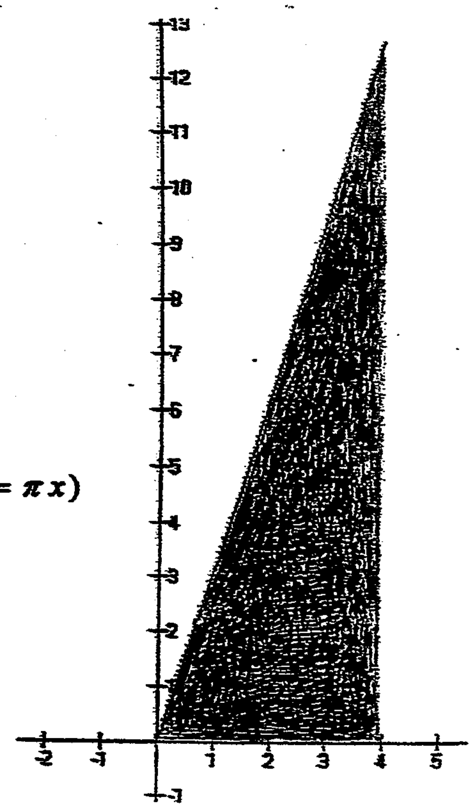


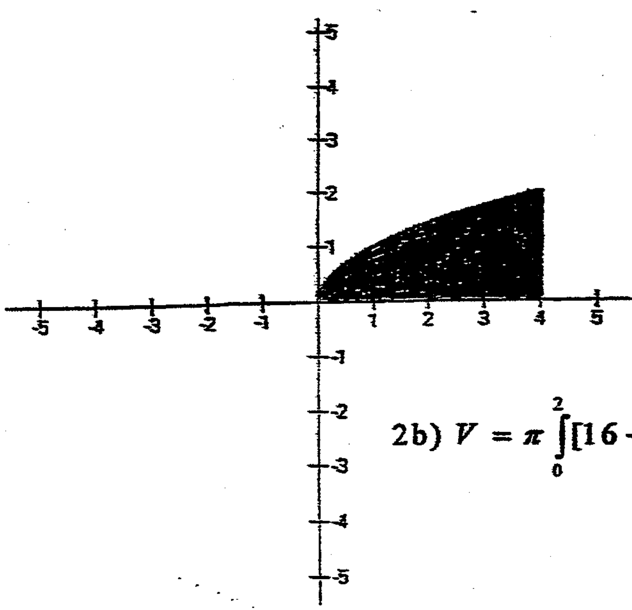
1c. $y = \sin x, y = 0, x = 2\pi, x = 3\pi$; about the y-axis.

$$2a.\ V = \pi \int_0^4 (\sqrt{x})^2 dx \quad (r(x) = \sqrt{x})$$



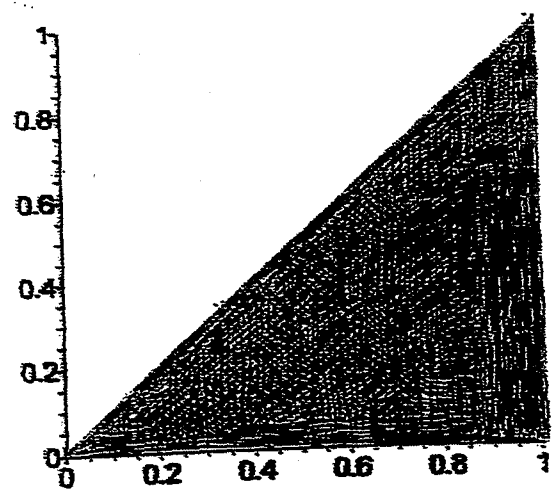
$$2a'.\ A = \int_0^4 \pi x dx \qquad (f(x) = \pi x)$$



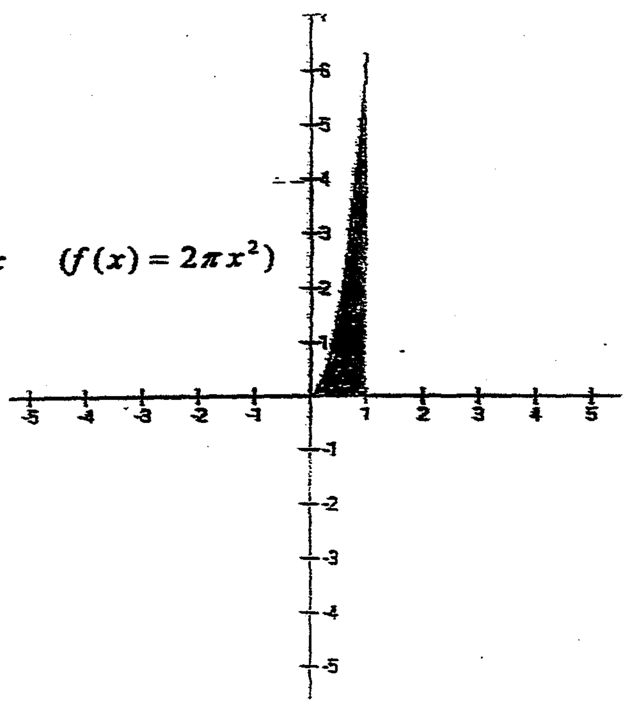


$$2b)\ V = \pi \int_0^2 [16 - (y^2)^2] dy \quad (R(x) = 4, r(x) = y^2)$$

2c) $V=2\pi\int_0^1 x^2dx \qquad (h(x) = x, r(x) = x)$



2c'. $A = \int_0^1 2\pi x^2 dx \qquad (f(x) = 2\pi x^2)$



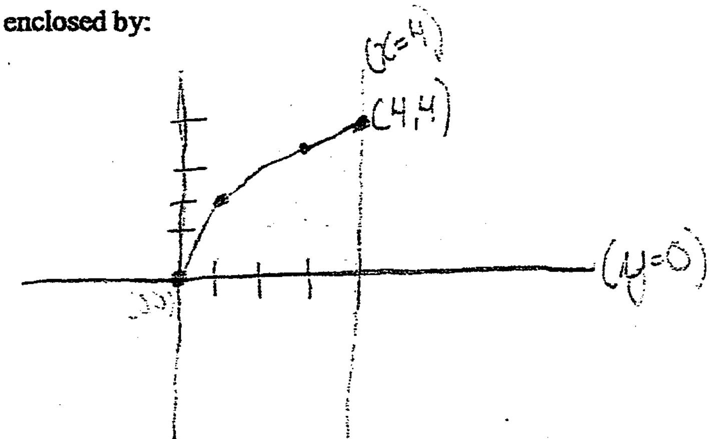
APPENDIX E

Examples of Students' Work

Exam

1. **Make a sketch of the region enclosed by:**

$y = 2\sqrt{x}, x = 4, y = 0$



2. **Use cylindrical shells to find the volume of the solid that results when the region enclosed by the equations in problem #1 is rotated around the y-axis.**

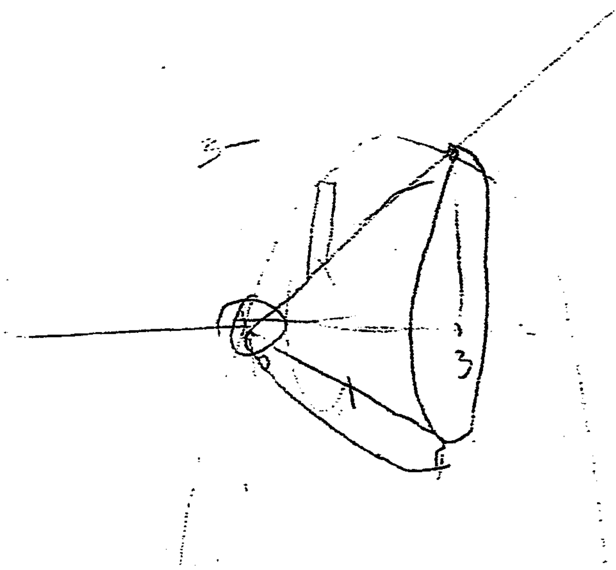
$y = 2\sqrt{x}$ $\quad x=4 \quad y=0$

$f(x) = 2\sqrt{x}$ $\quad x=4$ $\quad$ Intersection point: (4,4)

$A = \int_0^4 (2\sqrt{x}) = \frac{2\sqrt{x}^2}{2} = x$

$\boxed{\text{Area} = 4}$

Susan
Interview 1



$$\pi\int_0^3 (R^2 - r^2)\,dx$$

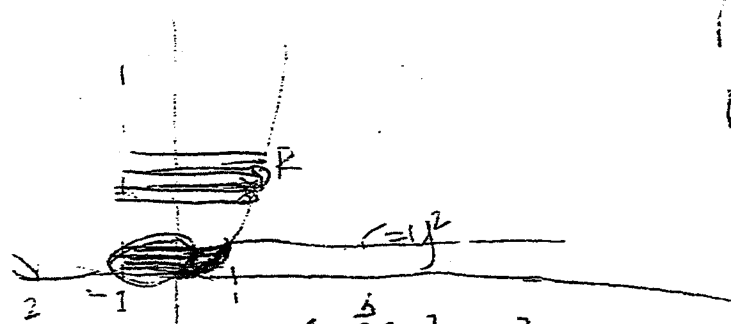
X

$4x - x^2$

$$\pi\int_0^3 \left[(4x-x^2)^2 - x^2\right] dx$$

$\sqrt{x} = \sqrt{y^2}$

$\sqrt{y} = y$



$$\pi\int (\sqrt{y} - y^2)\,dy$$

$$V = \pi\int_a^b \left(R^2(x) - r^2(x)\right) dy$$

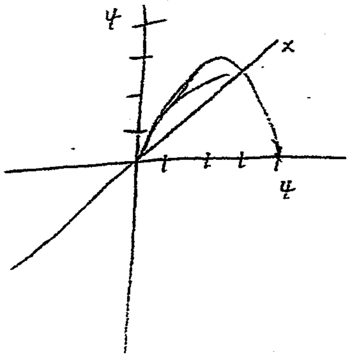
$$R(y) = \pi\int_0^1 \left[(\sqrt{y}+1)^2 - (y^2+1)^2\right] dy$$

$y + 1 - y^4 + 1$

$(\sqrt{y}+1)(\sqrt{y}+1)$

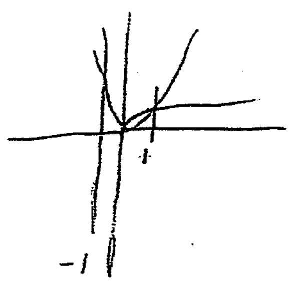
$$\pi\int \left[y + 2\sqrt{y} + 1\right] + \left[y^4 - 2y^2 \ldots\right.$$

Paul
Interview 1

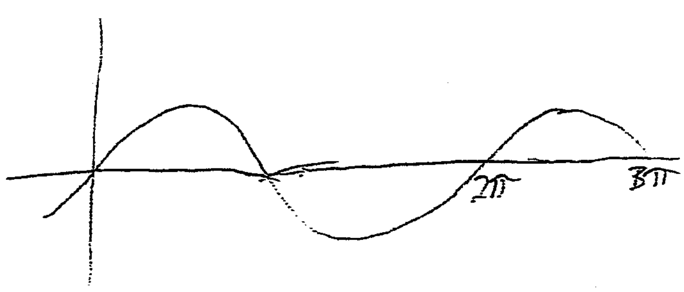


$$\pi\int_0^3 (4x-x^2)^2-(x)^2\,dx$$

$y=\sqrt{x}$
$y=x^2$

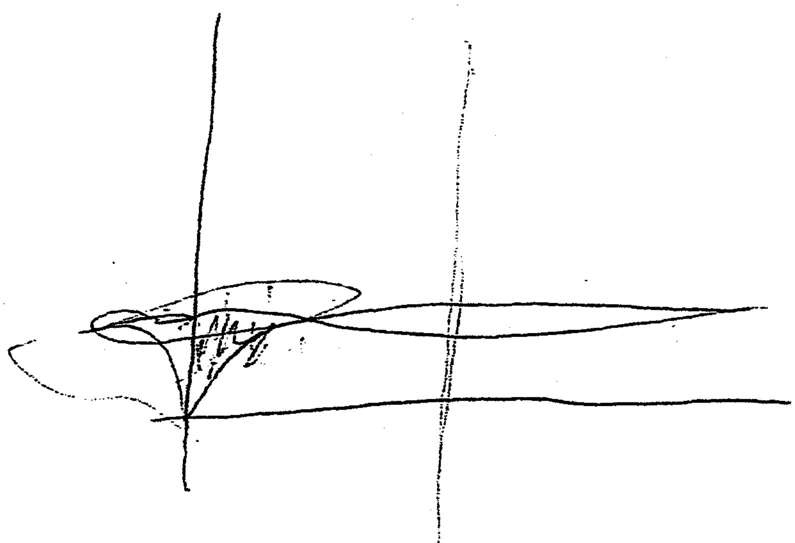


$$2\pi\int_0^1 x+1(\sqrt{x}-x^2)\,dx$$



$$2\pi\int_{2\pi}^{3\pi} x\sin x\,dx$$

Tom Interview 2



$16-(y^2)^2$ $\quad 16-y^4=x$

$x-16=y^4$ $\quad y=(x-16)^{1/4}$

$y=\sqrt{x-16}$

APPENDIX F

IRB Approval and Consent Forms

UNIVERSITY *of* NEW HAMPSHIRE

October 18, 2005

Mariana Montiel
Mathematics, Nesmith Hall
Durham, NH 03824

IRB #: 3460
Study: The Process of Integration and the Concept of Integral: How Does Success with Applications and Comprehension of Underlying Notions such as Accumulation Relate to Mathematical Fluency?
Approval Date: 07/18/2005

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Exempt as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 101(b). Approval is granted to conduct your study as described in your protocol.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* (This document is also available at http://www.unh.edu/osr/compliance/irb.html.) Please read this document carefully before commencing your work involving human subjects.

Upon completion of your study, please complete the enclosed pink Exempt Study Final Report form and return it to this office along with a report of your findings.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie Simpson

Julie F. Simpson
Manager

cc: File
Karen Graham

Research Conduct and Compliance Services, Office of Sponsored Research, Service Building, 51 College Road, Durham, NH 03824-3585 * Fax: 603-862-3564

UNIVERSITY *of* NEW HAMPSHIRE

CONSENT FORM FOR PARTICIPATION IN A RESEARCH STUDY

TITLE OF RESEARCH STUDY

The Process of Integration and the Concept of the Integral: How Does Success with Applications and Comprehension of Underlying Notions such as Accumulation Relate to Students' Mathematical Fluency.

I am a doctoral student at the University of New Hampshire.

WHAT IS THE PURPOSE OF THIS STUDY?

The purpose of this research is to clarify how students understand and use the integral in their second calculus course, and how this understanding affects their success in applying the techniques and realizing the applications they are expected to do.

WHAT DOES YOUR PARTICIPATION IN THIS STUDY INVOLVE?

- Your participation in this study will include between 3 and 5 sessions in which you will answer questions related to the material you are seeing in class, will try to solve problems, explaining to me your thinking as you go along. These sessions should take about an hour.
- These sessions should take about an hour.

WHAT ARE THE POSSIBLE RISKS OF PARTICIPATING IN THIS STUDY?

There are no risks.

WHAT HAPPENS IF I GET SICK OR HURT FROM TAKING PART IN THIS STUDY?

Not applicable.

You understand that if you are injured or require medical treatment, you may seek treatment from your primary care provider or, if eligible, from University Health Services. If you have paid a student-health fee, you will not be billed by University Health Services for services covered by that fee. If you have not paid the fee, you will be charged for services rendered by University Health Services. The University of New Hampshire is not responsible for the cost of any care required as a result of your participation in this study.

WHAT ARE THE POSSIBLE BENEFITS OF PARTICIPATING IN THIS STUDY?

As you will have some special time and attention, it is possible that you might gain some depth of understanding by participating in the sessions.

IF YOU CHOOSE TO PARTICIPATE IN THIS STUDY, WILL IT COST YOU ANYTHING?

No, it will not cost you anything.

WILL YOU RECEIVE ANY COMPENSATION FOR PARTICIPATING IN THIS STUDY?

Yes, you will receive $20 for the hour.

WHAT OTHER OPTIONS ARE AVAILABLE IF YOU DO NOT WANT TO TAKE PART IN THIS STUDY?

You understand that your consent to participate in this research is entirely voluntary, and that your refusal to participate will involve no prejudice, penalty or loss of benefits to which you would otherwise be entitled. Your refusal to participate will, by no means, affect your grade or class standing.

CAN YOU WITHDRAW FROM THIS STUDY?

If you consent to participate in this study, you are free to stop your participation in the study at any time without prejudice, penalty, or loss of benefits to which you would otherwise be entitled

HOW WILL THE CONFIDENTIALITY OF YOUR RECORDS BE PROTECTED?

The researcher seeks to maintain the confidentiality of all data and records associated with your participation in this research.

You should understand, however, there are rare instances when the researcher is required to share personally-identifiable information (e.g., according to policy, contract, regulation). For example, in response to a complaint about the research, officials at the University of New Hampshire, designees of the sponsor(s), and/or regulatory and oversight government agencies may access research data.

All information will be confidential, and the results will be reported with pseudonyms. The information transcribed during the interview sessions will be kept securely in a locked portfolio that only the researcher has access to.

WHOM TO CONTACT IF YOU HAVE QUESTIONS ABOUT THIS STUDY

If you have any questions pertaining to the research you can contact:

Mariana Montiel
mmontiel@cisunix.unh.edu
or

Karen Graham
karen.graham@unh.edu

to discuss them.

If you have questions about your rights as a research subject you can contact Julie Simpson in the UNH Office of Sponsored Research, 603-862-2003 or Julie.simpson@unh.edu to discuss them.

I, ______________________ CONSENT/AGREE to participate in this research study

______________________________ ______________
Signature of Subject Date